8. Local L-functions: the archimedean case

When $k = \mathbb{R}$ or $\mathbb{C}$ we still have our family of local integrals

$$\{\Psi(s, W, W')\} \quad \text{or} \quad \{\Psi_j(s, W, W')\} \quad \text{or} \quad \{\Psi(s, W, W', \Phi)\}$$

for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$, and $\Phi \in \mathcal{S}(k^n)$, now for π and π' irreducible admissible generic representations of $GL_n(k)$ or $GL_m(k)$ which are smooth and of moderate growth. In the current state of affairs the local L-functions $L(s, \pi \times \pi')$ are not defined intrinsically through the integrals, but rather extrinsically through the arithmetic Langlands classification and then related to the integrals.

8.1. **The arithmetic Langlands classification.** Both $k = \mathbb{R}$ and $k = \mathbb{C}$ have attached to them Weil groups W_k which play a role in their local class field theory similar to that of the richer $Gal(\overline{k}/k)$ for non-archimedean k.

When $k = \mathbb{C}$, $W_{\mathbb{C}} = \mathbb{C}^\times$ is simply the multiplicative group of $\mathbb{C}$. The only irreducible representations of $W_{\mathbb{C}}$ are thus characters.

When $k = \mathbb{R}$ then $W_{\mathbb{R}}$ can be defined as $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ where $jzj^{-1} = \overline{z}$ and $j^2 = -1 \in \mathbb{C}^\times$. This is an extension of $Gal(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^\times = W_{\mathbb{C}}$.

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{R}} \longrightarrow Gal(\mathbb{C}/\mathbb{R}) \longrightarrow 1$$

Now $W_{\mathbb{R}}$ has both one and two dimensional irreducible representations. Note that $W_{\mathbb{R}}^{ab} \simeq \mathbb{R}^\times$.

In rough terms, the arithmetic Langlands classification says there are natural bijections between

$$\mathcal{A}_n(k) = \{ \text{ irreducible admissible } \mathcal{H} - \text{modules} \quad \text{for} \quad GL_n(k)\}$$

and

$$\mathcal{G}_n(k) = \{n - \text{dimensional, semisimple representations of } W_k\}.$$

On the other hand, if

$$\mathcal{A}_n^\infty(k) = \{ \text{ irreducible admissible smooth moderate growth representations of } GL_n(k)\}$$

then the work of Casselman and Wallach gives a bijection between $\mathcal{A}_n(k)$ and $\mathcal{A}_n^\infty(k)$. Combining these, we can view the arithmetic Langlands classification as giving a natural bijection

$$\begin{array}{ccccc} \mathcal{A}_n^\infty(k) & \stackrel{\sim}{\longleftrightarrow} & \mathcal{A}_n(k) & \stackrel{\sim}{\longleftrightarrow} & \mathcal{G}_n(k) \\ \pi & & \longrightarrow & & \tau = \tau(\pi). \\ \pi = \pi(\tau) & & \longleftarrow & & \tau \end{array}$$

For example:

- If $\dim(\tau) = 1$, then $\pi(\tau)$ is a character of $GL_1(k)$.

- If $k = \mathbb{R}$ and τ is irreducible, unitary, and $\dim(\tau) = 2$, then $\pi(\tau)$ is a unitary discrete series representation of $GL_2(\mathbb{R})$.

- If $\tau = \bigoplus_{i=1}^{r} \tau_i$ with each τ_i irreducible, then $\pi(\tau)$ is the Langlands quotient of $Ind_Q^{GL_n}(\pi(\tau_1) \otimes \cdots \otimes \pi(\tau_r))$.

- If in addition π is generic, then

$$\pi = \pi(\tau) = Ind_Q^{GL_n}(\pi(\tau_1) \otimes \cdots \otimes \pi(\tau_r))$$

is a full irreducible induced representation from characters of $GL_1(k)$ and possible discrete series representations of $GL_2(k)$ if $k = \mathbb{R}$. (This result is due to Vogan and is not part of the classification per se.)

8.2. **The L-functions.** Set

$$\Gamma_k(s) = \begin{cases} \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) & k = \mathbb{R} \\ 2(2\pi)^{-s}\Gamma(s) & k = \mathbb{C} \end{cases}.$$

Then Weil attached to each semi-simple representation τ of W_k an L-function: $L(s, \tau)$. For example:

- If τ is an unramified character of $W_k^{ab} = k^\times$, say $\tau(x) = |x|_k^r$, then $L(s, \tau) = \Gamma_k(s + r)$.

- If $\dim(\tau) = 2$ and $\pi(\tau)$ is the holomorphic discrete series of weight k for $GL_2(\mathbb{R})$, then $L(s, \tau) = \Gamma_\mathbb{C}\left(s + \frac{k-1}{2}\right)$.

- If $\tau = \bigoplus_{i=1}^{r} \tau_i$ with each τ_i irreducible, then $L(s, \tau) = \prod_{i=1}^{r} L(s, \tau_i)$.

He also attached local ε-factors. For example, if τ is the character $\tau(x) = x^{-N}|x|_k^t$ and ψ is the standard additive character of k, then $\varepsilon(s,\tau,\psi) = i^N$.

There are natural "twisted" L-functions and ε-factors in this context, for if τ is an n-dimensional representation of W_k and τ' in a m-dimensional representation of W_k then the tensor product $\tau\otimes\tau'$ is an mn-dimensional representation and we have thus defined $L(s,\tau\otimes\tau')$ and $\varepsilon(s,\tau\otimes\tau',\psi)$ as well.

Now return to our representations π of $GL_n(k)$ and π' of $GL_m(k)$. Suppose that under the arithmetic Langlands classification we have $\pi = \pi(\tau)$ and $\pi' = \pi(\tau')$ with τ and τ' n-dimensional and $m-$dimensional representations of W_k respectively. Then we **define** the L-function for π and π' through the classification:

$$L(s,\pi\times\pi') = L(s,\tau\otimes\tau')$$
$$\varepsilon(s,\pi\times\pi',\psi) = \varepsilon(s,\tau\otimes\tau',\psi)$$

and we set

$$\begin{aligned}\gamma(s,\pi\times\pi',\psi) &= \frac{\varepsilon(s,\pi\times\pi',\psi)L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}{L(s,\pi\times\pi')}\\ &= \frac{\varepsilon(s,\tau\otimes\tau',\psi)L(1-s,\widetilde{\tau}\otimes\widetilde{\tau}')}{L(s,\tau\otimes\tau')}.\end{aligned}$$

Note that $L(s,\pi\times\pi')$ is always an archimedean Euler factor of degree nm.

8.3. **The integrals ($m < n$).** We now have to prove that this definition of the L-function behaves well with respect to our integrals. To analyze the integrals, we begin again with the properties of the Whittaker functions.

Proposition 8.1. *Let π be an irreducible admissible generic representation of $GL_n(k)$ which is smooth of moderate growth. Then there is a finite set of A-finite functions on A, say $X(\pi) = \{\chi_i\}$, depending only on π, such that for every $W \in \mathcal{W}(\pi,\psi)$ there exist Schwartz functions $\phi_i \in \mathcal{S}(k^{n-1}\times K)$ such that for $a \in A$ with $a_n = 1$ and $k \in K$ we have*

$$W\left(\begin{pmatrix} a_1 & & & \\ & \ddots & & \\ & & a_{n-1} & \\ & & & 1\end{pmatrix} k\right) = \sum_{X(\pi)} \chi_i(a)\phi_i(\alpha_1(a),\dots,\alpha_{n-1}(a);k).$$

As in the non-archimedean case, the A-finite functions in $X(\pi)$ are related to the archimedean Jacquet module of π and then through the classification to the associated representation τ of W_k. This then gives the same convergence estimates as before.

Proposition 8.2. *Each local integral $\Psi_j(s,W,W')$ converges absolutely for $Re(s) \gg 0$, and if π and π' are both unitary they converge absolutely for $Re(s) \geq 1$.*

The non-archimedean statements on rationality and "bounded denominators" are replaces by the following analysis.

Let $\mathcal{M}(\pi\times\pi') = \mathcal{M}(\tau\otimes\tau')$ be the space of all meromorphic functions $\phi(s)$ satisfying:

- If $P(s) \in \mathbb{C}[s]$ is a polynomial such that $P(s)L(s,\pi\times\pi')$ is holomorphic in the vertical strip $S[a,b] = \{s \mid a \leq Re(s) \leq b\}$. then $P(s)\phi(s)$ is holomorphic and bounded in $S[a,b]$.

As an exercise, one can show that $\phi \in \mathcal{M}(\pi\times\pi')$ implies that the ratio $\dfrac{\phi(s)}{L(s,\pi\times\pi')}$ is entire and bounded in vertical strips.

Theorem 8.1. *The integrals $\Psi_j(s,W,W')$ extend to meromorphic functions of s and as such $\Psi_j(s,W,W') \in \mathcal{M}(\pi\times\pi')$. In particular, the ratios $e_j(s,W,W') = \dfrac{\Psi_j(s,W,W')}{L(s,\pi\times\pi')}$ are entire and bounded in vertical strips.*

This is more than just "bounded denominators" since it specifies $L(s,\pi\times\pi')$ as a common denominator.

The same formal manipulations as in the non-archimedean show that if we set

$$\mathcal{I}_j(\pi\times\pi') = \langle \Psi_j(s,W,W') \mid W \in \mathcal{W}(\pi,\psi), W' \in \mathcal{W}(\pi',\psi^{-1})\rangle$$

then $\mathcal{I}_j(\pi\times\pi') = \mathcal{I}_{j+1}(\pi\times\pi')$ and hence $\mathcal{I}(\pi\times\pi') = \mathcal{I}_j(\pi\times\pi')$ is independent of j and

$$\mathcal{I}(\pi\times\pi') \subset \mathcal{M}(\pi\times\pi').$$

There is also a local functional equation, but unlike the non-archimedean case, the "factor of proportionality" $\gamma(s,\pi\times\pi',\psi)$ is specified a priori.

Theorem 8.2. *We have the local functional equation*

$$\widetilde{\Psi}(1-s, R(w_{n,m})\widetilde{W}, \widetilde{W}') = \omega_{\pi'}(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi(s,W,W')$$

with $\gamma(s,\pi\times\pi',\psi)=\gamma(s,\tau\otimes\tau',\psi)$.

The proofs of Theorems 8.1 and 8.2 are due to Jacquet and Shalika. Their strategy is roughly as follows:

(i) Very interestingly, they essentially show that Theorem 8.2 (the local functional equation) *implies* Theorem 8.1 (that the L-functions is essentially the correct denominator). This takes place in the space $\mathcal{M}(\pi\times\pi')$

(ii) If $m=1$, so π' is a character, they reduce Theorem 8.2 to previous results of Godement and Jacquet on standard L-functions for GL_n, which in turn reduced to the cases of $GL_2\times GL_1$ and $GL_1\times GL_1$ in that context.

(iii) If $m=2$ and π' is a discrete series representation of $GL_2(\mathbb{R})$, then they embed $\pi'\subset Ind(\mu_1\otimes\mu_2)$ and then reduce to (ii).

(iv) If $m>2$ and $\pi'=Ind(\pi_1'\otimes\cdots\otimes\pi_r')$ with each π_i' either a character or discrete series representation then they use a "multiplicativity" argument to again reduce to (ii) or (iii).

8.4. **Is the L-factor correct?** We know that $\mathcal{I}(\pi\times\pi')\subset\mathcal{M}(\pi\times\pi')$, so that $L(s,\pi\times\pi')=L(s,\tau\otimes\tau')$ contains all poles of our local integrals. We are left with the following two related questions.

1. Is $L(s,\pi\times\pi')$ the minimal such factor?

2. Can we write
$$L(s,\pi\times\pi')=\sum_{i=1}^{r}\Psi(s,W_i,W_i')$$
as a finite linear combination of local integrals?

To investigate these questions, Jacquet and Shalika had to first enlarge the family of local integrals. If Λ and Λ' are continuous Whittaker functionals on V_π and $V_{\pi'}$ then their tensor product $\hat{\Lambda}=\Lambda\otimes\Lambda'$ is a continuous linear functional on the algebraic tensor product $V_\pi\otimes V_{\pi'}$ which extends continuously to the topological tensor product $V_{\pi\hat{\otimes}\pi'}=V_\pi\hat{\otimes}V_{\pi'}$. (Note that this completion is in fact the Casselman-Wallach canonical completion of the algebraic tensor product. So to remain categorical, this is natural.) Then for $\xi\in V_\pi\hat{\otimes}V_{\pi'}$ we can define
$$W_\xi(g,h)=\hat{\Lambda}(\pi(g)\otimes\pi'(h)\xi),$$

so that $W_\xi \in \mathcal{W}(\pi\hat{\otimes}\pi') = \mathcal{W}(\pi,\psi)\hat{\otimes}\mathcal{W}(\pi',\psi^{-1})$, and then

$$\Psi(s,W) = \int_{N_m(k)\backslash GL_m(k)} W_\xi\left(\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}, h\right) |\det(h)|^{s-\frac{n-m}{2}}\, dh.$$

Essentially the same arguments as before give Theorems 8.1 and 8.2 for these extended integrals. If we set

$$\mathcal{I}(\pi\hat{\otimes}\pi') = \langle \Psi(s,W) \mid W \in \mathcal{W}(\pi\hat{\otimes}\pi')\rangle$$

then again we have $\mathcal{I}(\pi\hat{\otimes}\pi') \subset \mathcal{M}(\pi\times\pi')$. But now they are able to show that in fact these spaces are equal.

Theorem 8.3. $\mathcal{I}(\pi\hat{\otimes}\pi') = \mathcal{M}(\pi\times\pi')$.

So $L(s,\pi\times\pi')$ is the correct denominator for the extended family $\mathcal{I}(\pi\hat{\otimes}\pi')$. This partially answers our first question. We also obtain a partial answer to our second question.

Corollary 8.3.1. *There exists* $W \in \mathcal{I}(\pi\hat{\otimes}\pi')$ *such that* $\Psi(s,W) = L(s,\pi\times\pi')$.

In order to investigate our questions for our original family, with Piatetski-Shapiro we showed the following continuity result.

Proposition 8.3. *The functionals*

$$W \mapsto e(s,W) = \frac{\Psi(s,W)}{L(s,\pi\times\pi')}$$

is continuous on $\mathcal{W}(\pi\hat{\otimes}\pi')$*, uniformly for* s *in compact subsets.*

Since the algebraic tensor product $\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\pi',\psi^{-1})$ is dense in $\mathcal{W}(\pi\hat{\otimes}\pi')$ and by the above corollary there exists $W \in \mathcal{W}(\pi\hat{\otimes}\pi')$ with $e(s,W)\equiv 1$ we then obtain the following result.

Corollary 8.3.2. *For each* $s_0 \in \mathbb{C}$ *there exist* $W \in \mathcal{W}(\pi,\psi)$ *and* $W' \in \mathcal{W}(\pi',\psi^{-1})$ *such that*

$$e(s_0,W,W') = \frac{\Psi(s,W,W')}{L(s,\pi\times\pi')} \neq 0.$$

Moreover, one can take W *and* W' *to be* K*-finite Whittaker functions.*

So $L(s,\pi\times\pi')$ is precisely the archimedean Euler factor of degree nm determined by the poles of original family of integrals $\mathcal{I}(\pi\times\pi')$. This finally answers question 1.

As for question 2, the answer is more ambiguous. There are definitive results only in the cases of $m = n$ and $m = n - 1$. In the case where π and π' are both unramified, Stade has done the archimedean unramified calculation.

Theorem 8.4. *If $n = m$ or $n = m-1$ and both π and π' are unramified then*

$$L(s, \pi \times \pi') = \begin{cases} \Psi(s, W^\circ, W'^\circ, \Phi^\circ) & m = n \\ \Psi(s, W^\circ, W'^\circ) & m = n - 1 \end{cases}$$

where W°, W'°, and Φ° are all normalized and unramified.

This has been generalized by Jacquet and Shalika, utilizing the last corollary.

Theorem 8.5. *If $m = n$ or $m = n - 1$ then there are finite collections of K-finite Whittaker functions $W_i \in \mathcal{W}(\pi, \psi)$ and $W'_i \in \mathcal{W}(\pi', \psi^{-1})$ and possibly $\Phi_i \in \mathcal{S}(k^n)$ such that*

$$L(s, \pi \times \pi') = \begin{cases} \sum_i \Psi(s, W_i, W'_i, \Phi_i) & m = n \\ \\ \sum_i \Psi(s, W_i, W'_i) & m = n - 1 \end{cases}.$$

It is somewhat widely believed that this last result will not extend to $m \leq n - 2$, even if one relaxes the K-finiteness condition.

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