7. The unramified calculation

In this lecture I would like to calculate $L(s, \pi \times \pi')$ when both π and π' are unramified, that is, they both have vectors fixed under their respective maximal compact subgroups $GL_n(\mathcal{O})$ and $GL_m(\mathcal{O})$. We will do this by explicitly computing the local integral $\Psi(s, W^\circ, W'^\circ)$ for W° and W'° the normalized K-fixed Whittaker functions. This calculation is similar to and motivated by the calculation of the p-Euler factor for $L(s, f)$ for f a classical cusp form. Recall that in the classical case of $f \in S_k(SL_2(\mathbb{Z}))$ to be able to compute the p-Euler factor for $L(s, f)$ we needed to know two things:

(i) that f was an eigen-function for all Hecke operators T_p or T_n;
(ii) the recursion among the T_{p^r} for a fixed p.

Now again let k be a non-archimedean local field of characteristic 0 with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{p}$ and uniformizer ϖ. Let

$$\mathcal{H}_K = \mathcal{H}(GL_n(k)//K) = C_c^\infty(GL_n(k)//GL_2(\mathcal{O}))$$

be the spherical Hecke algebra for $GL_n(k)$ consisting of compactly supported functions on $GL_n(k)$ which are bi-K-invariant, as in Lecture 3. This plays the role of the classical Hecke algebra in this context. It is a convolution algebra as before. For each i, $0 \leq i \leq n$, let Φ_i be the characteristic function

$$\Phi_i = Char\left(GL_n(\mathcal{O}) \begin{pmatrix} \varpi I_i & \\ & I_{n-i} \end{pmatrix} GL_n(\mathcal{O})\right)$$

so that ϖ occurs in the first i diagonal entries. (For $G = GL_2$ and $k = \mathbb{Q}_p$, Φ_1 is the avatar of the classical Hecke operator T_p.) Then a standard fact is:

Proposition 7.1. *The spherical Hecke algebra $\mathcal{H}_K$ is a commutative algebra and is generated by the Φ_i for $1 \leq i \leq n$.*

For any smooth representation (π, V_π) of $GL_n(k)$ we have an action of $\mathcal{H}$ or $\mathcal{H}_K$ on V_π as a convolution algebra via

$$\pi(\Phi)v = \int_{GL_n(k)} \Phi(g)\pi(g)v \, dg.$$

Note that since π is smooth and Φ has compact support, this is really a finite sum. In the transition from classical modular forms to automorphic representations and back, this corresponds to the action of the classical Hecke operators on modular forms.

7.1. **Unramified representations.** Now let (π, V_π) be an irreducible admissible smooth generic representation of $GL_n(k)$ which is unramified. Then it is known that

$$\pi = Ind_{B(k)}^{GL_n(k)}(\mu_1 \otimes \cdots \otimes \mu_n)$$

is a full induced representation from the Borel subgroup $B(k)$ of unramified characters μ_i of $k^\times$. Here unramified means that each μ_i is invariant under the maximal compact subgroup $\mathcal{O}^\times \subset k^\times$. Since $k^\times = \coprod \varpi^j \mathcal{O}^\times$, each character μ_i is completely determined by its value $\mu_i(\varpi) \in \mathbb{C}^\times$. Thus in turn π will be completely determined by the n complex numbers

$$\{\mu_1(\varpi), \ldots, \mu_n(\varpi)\}$$

which can be encoded in a diagonal matrix

$$A_\pi = \begin{pmatrix} \mu_1(\varpi) & & \\ & \ddots & \\ & & \mu_n(\varpi) \end{pmatrix} \in GL_n(\mathbb{C}).$$

These parameters, whether viewed as n non-zero complex numbers, the matrix $A_\pi \in GL_n(\mathbb{C})$ or the conjugacy class $[A_\pi] \subset GL_n(\mathbb{C})$ are the *Satake parameters* of the unramified representation π.

Since (π, V_π) is unramified, then there is a unique (up to scalar multiples) non-zero K-fixed vector $v^\circ \in V_\pi$. If $\Phi \in \mathcal{H}_K$, the spherical Hecke algebra, then $\pi(\Phi)v^\circ$ will again be K-fixed. Thus we obtain

$$\pi(\Phi)v^\circ = \Lambda_\pi(\Phi)v^\circ$$

with $\Lambda_\pi : \mathcal{H}_K \to \mathbb{C}$ a character of $\mathcal{H}_K$ as a convolution algebra. Thus v° is our local Hecke eigen-function.

For $\pi = Ind(\mu_1 \otimes \cdots \otimes \mu_n)$ it is easy to compute this character on the generators Φ_i of $\mathcal{H}_K$. As in the classical case, we will need to know how to decompose the associated double coset into single cosets.

For each $J \in \mathbb{Z}^n$, say $J = (j_1, \ldots, j_n)$, let

$$\varpi^J = \begin{pmatrix} \varpi^{j_1} & & \\ & \ddots & \\ & & \varpi^{j_n} \end{pmatrix} \in GL_n(k).$$

So if we set $\eta_i = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ with the first i entries of 1 and the others 0, then Φ_i is the characteristic function of $K\varpi^{\eta_i}K$. To decompose this double coset into single ones, let us set

$$I_i = \left\{\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{Z}^n \mid \epsilon_j \in \{0, 1\}, \sum \epsilon_j = i\right\}$$

and for each $\epsilon \in I_i$ let

$$N(\mathcal{O}, \epsilon) = N(\mathcal{O}) \cap \varpi^{\epsilon} K \varpi^{-\epsilon}.$$

Lemma 7.1.

$$K\varpi^{\eta_i} K = \coprod_{\epsilon \in I_i} \coprod_{n \in N(\mathcal{O})/N(\mathcal{O},\epsilon)} n\varpi^{\epsilon} K.$$

Now let f° be the K-fixed vector in $Ind(\mu_1 \otimes \cdots \otimes \mu_n)$ normalized so that $f^{\circ}(e) = 1$. Then we have

$$(\pi(\Phi_i) f^{\circ})(e) = \Lambda(\Phi_i) f^{\circ}(e) = \Lambda_{\pi}(\Phi_i).$$

On the other hand, we can do the explicit computation in the induced model. By definition

$$f^{\circ}(nak) = \delta_{B_n}^{1/2}(a) \prod_{i=1}^{n} \mu_i(a_i) f^{\circ}(e) = \delta_B^{1/2}(a) \prod_{i=1}^{n} \mu_i(a_i)$$

for $n \in N_n(k)$, $a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in A_n(k)$, and $k \in K_n$. Then we can compute

$$\begin{aligned}
(\pi(\Phi_i) f^{\circ})(e) &= \int_{GL_n(k)} \Phi_i(g) f^{\circ}(g)\ dg = \int_{K\varpi^{\eta_i}K} f^{\circ}(g)\ dg \\
&= \sum_{\epsilon \in I_i} \sum_{n \in N(\mathcal{O})/N(\mathcal{O},\epsilon)} f(n\varpi^{\epsilon}) \\
&= \sum_{\epsilon \in I_i} |N(\mathcal{O})/N(\mathcal{O},\epsilon)| \delta_{B_n}^{1/2}(\varpi^{\epsilon}) \prod_{j=1}^{n} \mu_j(\varpi)^{\epsilon_j}.
\end{aligned}$$

An elementary computation then gives

$$|N(\mathcal{O})/N(\mathcal{O},\epsilon)| \delta_{B_n}^{1/2}(\varpi^{\epsilon}) = q^{i(n-i)/2}$$

so that

$$\begin{aligned}
(\pi(\Phi_i) f^{\circ})(e) &= q^{i(n-i)/2} \sum_{\epsilon \in I_i} \prod_{j=1}^{n} \mu_j(\varpi)^{\epsilon_j} \\
&= q^{i(n-i)/2} \sigma_i(\mu_1(\varpi), \ldots, \mu_n(\varpi))
\end{aligned}$$

where σ_i is the ith elementary symmetric polynomial in the $\mu_j(\varpi)$.

Comparing our two expressions for $(\pi(\Phi_i) f^{\circ})(e)$ we obtain

Proposition 7.2. *For $\pi = Ind(\mu_1 \otimes \cdots \otimes \mu_n)$ unramified*

$$\Lambda_\pi(\Phi_i) = q^{i(n-i)/2}\sigma_i(\mu_1(\varpi), \ldots, \mu_n(\varpi)).$$

This computes the Hecke eigen-values in terms of the Satake parameters of π.

7.2. **Unramified Whittaker functions.** The analogue of the classical recursion relation for the Hecke operators T_{p^r} can now be employed to compute a *formula* for the unramified Whittaker function W° that occurs in our integrals. For GL_n this was first done by Shintani, who we follow.

Take $\psi : k \to \mathbb{C}$ our additive character to also be unramified and non-trivial, so $\psi(\mathcal{O}) = 1$ but $\psi(\varpi^{-1}) \neq 1$. Let $W^\circ \in \mathcal{W}(\pi, \psi)$ be the K-fixed Whittaker function in $\mathcal{W}(\pi, \psi)$. By the Iwasawa decomposition, any $g \in GL_n(k)$ can be written

$$g = nak \in NAK \quad \text{with} \quad a = \varpi^J \in A$$

for some $J \in \mathbb{Z}^n$. Then

$$W^\circ(g) = W^\circ(n\varpi^J k) = \psi(n)W^\circ(\varpi^J).$$

So it suffices to compute the values $W^\circ(\varpi^J)$. The same calculation that gave the "rapid decrease" of W on A in the $GL_2(k)$ case now gives

$$W^\circ(\varpi^J) = 0 \quad \text{unless} \quad j_1 \geq j_2 \geq \cdots \geq j_n.$$

We next do an explicit calculation of the action of each $\Phi_i \in \mathcal{H}_K$ in the Whittaker model. We still have that

$$(\pi(\Phi_i)W^\circ)(\varpi^J) = \Lambda_\pi(\Phi_i)W^\circ(\varpi^J)$$

for all J, with an explicit formula for $\Lambda(\Phi_i)$. Computing in the Whittaker model we have

$$\begin{aligned}(\pi(\Phi_i)W^\circ)(\varpi^J) &= \int_{K\varpi^{\eta_i}K} W^\circ(\varpi^J g)\ dg \\ &= \sum_{\epsilon \in I_i} \sum_{n \in N(\mathcal{O})/N(\mathcal{O},\epsilon)} W^\circ(\varpi^J n \varpi^\epsilon) \\ &= \sum_{\epsilon \in I_i} \sum_{n \in N(\mathcal{O})/N(\mathcal{O},\epsilon)} \psi(\varpi^J n \varpi^{-J}) W^\circ(\varpi^{J+\epsilon}).\end{aligned}$$

Since $j_1 \geq \cdots \geq j_n$, we have that $\varpi^J n \varpi^{-J} \in N(\mathcal{O})$ so that the value of ψ on this element is 1. Hence

$$\begin{aligned}(\pi(\Phi_i)W^\circ)(\varpi^J) &= \sum_{\epsilon \in I_i} |N(\mathcal{O})/N(\mathcal{O},\epsilon)| W^\circ(\varpi^{J+\epsilon}) \\ &= \sum_{\epsilon \in I_i} \delta_{B_n}^{-1/2}(\varpi^\epsilon) q^{i(n-i)/2} W^\circ(\varpi^{J+\epsilon}).\end{aligned}$$

If we then combine our two expressions for $(\pi(\Phi_i)W^\circ)(\varpi^J)$ we obtain our recursion.

Proposition 7.3. *For the unramified Whittaker function in $\mathcal{W}(\pi,\psi)$ we have the recursion*

$$\Lambda_\pi(\Phi_i)W^\circ(\varpi^J) = q^{i(n-i)/2} \sum_{\epsilon \in I_i} \delta_{B_n}^{-1/2}(\varpi^\epsilon) W^\circ(\varpi^{J+\epsilon}).$$

The solution to this recursion is quite interesting. It involves the characters of finite dimensional representations of $GL_n(\mathbb{C})$. The n-tuples $J = (j_1, \ldots, j_n)$ with $j_1 \geq \cdots \geq j_n$ are the possible highest weights for the finite dimensional representations of $GL_n(\mathbb{C})$. Let ρ_J denote the finite dimensional representation of highest weight J and let $\chi_J = Tr(\rho_J)$ be its character. Then many things are known about these characters, for example, from the formula for the decomposition of the tensor product of two finite dimensional representations we also obtain a recursion

$$\chi_{\eta_i}\chi_J = \sum_{\epsilon \in I_i} \chi_{J+\epsilon}$$

similar to the recursion for $W^\circ(\varpi^J)$. Since the χ_J are class functions on $GL_n(\mathbb{C})$, it makes sense to evaluate them on our Satake class A_π for π. For example, since ρ_{η_i} is the ith exterior power of the standard representation of GL_n we find that

$$\chi_{\eta_i}(A_\pi) = \sigma_i(\mu_1(\varpi), \ldots, \mu_n(\varpi)) = q^{-i(n-i)/2}\Lambda_\pi(\Phi_i).$$

Utilizing these facts from finite dimensional representation theory it is then a simple matter to solve the recursion for the $W^\circ(\varpi^J)$ in terms of the $\chi_J(A_\pi)$ and obtain Shintani's formula.

Proposition 7.4. $W^\circ(\varpi^J) = \delta_{B_n}^{1/2}(\varpi^J)\chi_J(A_\pi)$.

7.3. **Calculating the integral.** We now return to our local integral. We consider the case $m < n$ and π, π', and ψ all unramified. Let $W^\circ \in \mathcal{W}(\pi, \psi)$ and $W'^\circ \in \mathcal{W}(\pi', \psi^{-1})$ be the normalized K-fixed Whittaker functions computed above. We have

$$\Psi(s, W^\circ, W'^\circ) = \int_{N_m(k)\backslash GL_m(k)} W^\circ \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} W'^\circ(h) |\det(h)|^{s-\frac{n-m}{2}}\, dh.$$

Use the Iwasawa decomposition to write $GL_m = N_m A_m K_m$ so that $h = n\varpi^J k$ and $dh = dn\ \delta_{B_m}^{-1}(\varpi^J)\ dk$. Then

$$\begin{aligned}\Psi(s,&W^\circ, W'^\circ)\\ &= \sum_{J\in\mathbb{Z}^m} W^\circ \begin{pmatrix} \varpi^J & \\ & I_{n-m} \end{pmatrix} W'^\circ(\varpi^J) |\det(\varpi^J)|^{s-\frac{n-m}{2}} \beta_{B_m}^{-1}(\varpi^J).\end{aligned}$$

Now $W'^\circ(\varpi^J) = 0$ unless $j_1 \geq \cdots \geq j_m$ and $W^\circ \begin{pmatrix} \varpi^J & \\ & I_{n-m} \end{pmatrix} = 0$ unless $j_1 \geq \cdots \geq j_m \geq 0$. Moreover, $|\det(\varpi^J)| = q^{-|J|}$ where $|J| = j_1 + \cdots + j_m$. So our integral becomes

$$\begin{aligned}\Psi(s,&W^\circ, W'^\circ)\\ &= \sum_{j_1\geq\cdots\geq j_m\geq 0} W^\circ \begin{pmatrix} \varpi^J & \\ & I_{n-m} \end{pmatrix} W'^\circ(\varpi^J) q^{-|J|(s-\frac{n-m}{2})} \beta_{B_m}^{-1}(\varpi^J).\end{aligned}$$

We next insert the formula from Proposition 7.4 and use the elementary fact that

$$\delta_{B_n}^{1/2} \begin{pmatrix} \varpi^J & \\ & I_{n-m} \end{pmatrix} \delta_{B_m}^{-1/2}(\varpi^J) = q^{-|J|\frac{n-m}{2}}$$

to obtain

$$\Psi(s, W^\circ, W'^\circ) = \sum_{j_1\geq\cdots\geq j_m\geq 0} \chi_{(J,0)}(A_\pi)\chi_J(A_{\pi'}) q^{-|J|s}$$

where $(J, 0) = (j_1, \ldots, j_m, 0, \ldots, 0)$ represents J filled out to be a vector in $\mathbb{Z}^n$.

We next use some fairly standard facts from the finite dimensional representation theory of $GL_n(\mathbb{C})$, namely

$$\chi_{(J,0)}(D_n)\chi_J(D_m) = Tr(\rho_{(J,0)}(D_n) \otimes \rho_J(D_m)),$$

$$\sum_{\substack{j_1\geq\cdots\geq j_m\geq 0\\ |J|=r}} Tr(\rho_{(J,0)}(D_n) \otimes \rho_J(D_m)) = Tr(S^r(D_n \otimes D_m)),$$

and

$$\sum_{r=0}^{\infty} Tr(S^r(D))X^r = \det(1 - XD)^{-1},$$

where $S^r(D)$ is the rth symmetric power of D. Applying these with $D_n = A_\pi$ and $D_m = A_{\pi'}$ finishes our calculation of the local integral.

Proposition 7.5. *If π, π', and ψ are all unramified, then*

$$\Psi(s, W^\circ, W'^\circ) = \det(I - q^{-s} A_\pi \otimes A_{\pi'})^{-1} = \prod_{i,j} (1 - \mu_i(\varpi)\mu'_j(\varpi)q^{-s})^{-1}.$$

Since $L(s, \pi \times \pi')$ is the minimal inverse polynomial in q^{-s} killing all poles of the family of local integrals this implies

$$\det(I - q^{-s} A_\pi \otimes A_{\pi'}) | L(s, \pi \times \pi')^{-1}.$$

Then comparing the poles of this factor with the potential poles coming from the asymptotics of W° and W'° as the simple roots go to zero from Lecture 6 gives us our result.

Theorem 7.1. *If π, π', and ψ are all unramified, then*

$$L(s, \pi \times \pi') = \det(I - q^{-s} A_\pi \otimes A_{\pi'})^{-1} = \Psi(s, W^\circ, W'^\circ).$$

Note that the degree of this Euler factor is mn. Moreover,

$$L(s, \pi) = \det(I - q^{-s} A_\pi)^{-1} = \prod (1 - \mu_i(\varpi)q^{-s})^{-1}$$

is an Euler factor of degree n. The same result holds for $GL_n \times GL_n$. One then takes for the Schwartz function $\Phi \in \mathcal{S}(k^n)$ the characteristic function Φ° of $\mathcal{O}^n \subset k^n$.

Since the factor $\varepsilon(s, \pi \times \pi', \psi)$ satisfies the local functional equation

$$\frac{\widetilde{\Psi}(1 - s, R(w_{n,m})\widetilde{W}^\circ, \widetilde{W'}^\circ)}{L(1 - s, \widetilde{\pi} \times \widetilde{\pi}')} = \omega_{\pi'}(-1)^{n-1} \varepsilon(s, \pi \times \pi', \psi) \frac{\Psi(s, W^\circ, W'^\circ)}{L(s, \pi \times \pi')}$$

we can conclude the following corollary.

Corollary 7.1.1. *If π, π', and ψ are all unramified, then*

$$\varepsilon(s, \pi \times \pi', \psi) \equiv 1.$$

In particular, taking π' to be the trivial character of GL_1 we see that if π is unramified then its conductor $f(\pi) = 0$.

Finally, as a second corollary we obtain the Jacquet-Shalika bounds on the Satake parameters.

Corollary 7.1.2. *Suppose that π is a irreducible unitary generic unramified representation of $GL_n(k)$, $\pi \simeq Ind(\mu_1 \otimes \cdots \otimes \mu_n)$. Then the Satake parameters $\mu_i(\varpi)$ satisfy*

$$q^{-1/2} < |\mu_i(\varpi)| < q^{1/2}.$$

To see this, we apply the $GL_n \times GL_n$ unramified calculation to π and $\pi' = \overline{\pi}$, the complex conjugate representation. Then $A_{\pi'} = A_{\overline{\pi}} = \overline{A_\pi}$ and

$$\det(I - q^{-s} A_\pi \otimes \overline{A_\pi}) \Psi(s, W^\circ, W'^\circ, \Phi^\circ) = 1.$$

The local integral is absolutely convergent for $Re(s) \geq 1$ since π is unitary. Then

$$\det(I - q^{-s} A_\pi \otimes \overline{A_\pi}) \neq 0 \quad \text{for} \quad Re(s) \geq 1.$$

This determinant has as a factor $(1 - |\mu_i(\varpi)|^2 q^{-s})$, so this also cannot vanish for $Re(s) \geq 1$. Hence

$$|\mu_i(\varpi)| < q^{1/2}.$$

Applying the same argument to $\widetilde{\pi}$ gives

$$|\mu_i(\varpi)|^{-1} < q^{1/2}$$

since $A_{\widetilde{\pi}} = A_\pi^{-1}$. Thus we have the result.

References

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