4. Fourier Expansions and Multiplicity One Theorems

We now start with results which are often GL_n *specific*. So we let $G = G_n = GL_n$ (however one should also keep in mind $G = GL_n \times GL_m$) and still take k to be a number field.

4.1. The Fourier expansion of a cusp form. Let (π, V_π) be a smooth cuspidal representation, so $V_\pi \subset \mathcal{A}_o^\infty$. Let $\varphi \in V_\pi$ be a smooth cusp form.

We begin with $G = GL_2$. Our translation subgroup is

$$N = N_2 = \left\{ n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}.$$

For any $g \in G(\mathbb{A})$ the function

$$x \mapsto \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right)$$

is a smooth function of $x \in \mathbb{A}$ which is periodic under k. Since $k\backslash\mathbb{A}$ is a compact abelian group we will have an abelian Fourier expansion of this function.

For each continuous character $\psi : k\backslash\mathbb{A} \to \mathbb{C}$ we define a ψ-Fourier coefficient, or ψ-Whittaker function, of φ by

$$W_{\varphi,\psi}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi^{-1}(x)\, dx.$$

This function satisfies

$$W_{\varphi,\psi}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \psi(x) W_{\varphi,\psi}(g).$$

Then by standard abelian Fourier analysis we have

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \sum_{\psi \in \widehat{k\backslash\mathbb{A}}} W_{\varphi,\psi}(g)\psi(x)$$

or

$$\varphi(g) = \sum_{\psi} W_{\varphi,\psi}(g).$$

By standard duality theory, $\widehat{k\backslash\mathbb{A}} \simeq k$ and is we *fix* one non-trivial character ψ then any other is of the form $\psi_\gamma(x) = \psi(\gamma x)$ for $\gamma \in k$, so

$$\varphi(g) = \sum_{\gamma \in k} W_{\varphi,\psi_\gamma}(g).$$

Since φ is cuspidal, for $\gamma = 0$ we have

$$W_{\varphi,\psi_0}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\, dx = 0$$

and for $\gamma \neq 0$ it is an easy change of variables to see that

$$W_{\varphi,\psi_\gamma}(g) = W_{\varphi,\psi}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

which gives for our Fourier expansion for GL_2

$$\varphi(g) = \sum_{\gamma\in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

where we have set $W_{\varphi,\psi} = W_\varphi$.

Now consider $G = GL_n$. The role of the translations is played by the full maximal unipotent subgroup

$$N = N_n = \left\{ n = \begin{pmatrix} 1 & x_{1,2} & & * \\ & \ddots & \ddots & \\ & & \ddots & x_{n-1,n} \\ 0 & & & 1 \end{pmatrix} \right\}$$

which is now non-abelian. If we retain out fixed additive character ψ of $k\backslash\mathbb{A}$ from before, then ψ defines a (continuous) character of $N(k)\backslash N(\mathbb{A})$ by

$$\psi(n) = \psi\left(\begin{pmatrix} 1 & x_{1,2} & & * \\ & \ddots & \ddots & \\ & & \ddots & x_{n-1,n} \\ 0 & & & 1 \end{pmatrix}\right) = \psi(x_{1,2} + \cdots + x_{n-1,n}).$$

The associated ψ-Whittaker function of φ is now

$$W_\varphi(g) = W_{\varphi,\psi}(g) = \int_{N(k)\backslash N(\mathbb{A})} \varphi(ng)\psi^{-1}(n)\, dn$$

which again satisfies $W_\varphi(ng) = \psi(n)W_\varphi(g)$ for all $n \in N(\mathbb{A})$. The Fourier expansion of φ which is useful is

$$\varphi(g) = \sum_{\gamma\in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

This is not hard to prove. It is essentially an induction based on the above argument, begun by expanding about the last column of N, which is abelian.

For $G = GL_3$ one would begin with

$$\varphi\left(\begin{pmatrix}1 & x_1 & x_2\\ & 1 & x_3\\ & & 1\end{pmatrix} g\right) = \varphi\left(\begin{pmatrix}1 & & x_2\\ & 1 & x_3\\ & & 1\end{pmatrix}\begin{pmatrix}1 & x_1 & \\ & 1 & \\ & & 1\end{pmatrix} g\right)$$

and expand this as a function of $\begin{pmatrix}x_2\\ x_3\end{pmatrix} \in (k\backslash\mathbb{A})^2$. Remember that $((k\backslash\mathbb{A})^2)^\wedge \simeq k^2$ and that $GL_2(k)$ acts on k^2 with two orbits: $\{0\}$ and an open orbit $(0,1)\cdot GL_2(k)$. The $\{0\}$ orbit contributes 0 by cuspidality and the open orbit can be parameterized by $P_2(k)\backslash GL_2(k)$ where $P_2 = \left\{\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\right\} = Stab((0,1))$. One then expands the resulting terms as functions of x_1 as before.

As I said, the proof is not hard. The difficult thing, if there is one, is in recognizing that this is what one needs. This was recognized independently by Piatetski-Shapiro and Shalika.

4.2. **Whittaker models.** Consider now the functions $W = W_\varphi$ which appear in the Fourier expansion of our cusp forms $\varphi \in V_\pi$. These are smooth functions on $G(\mathbb{A})$ satisfying $W(ng) = \psi(n)W(g)$ for all $n \in N(\mathbb{A})$. Let

$$\mathcal{W}(\pi,\psi) = \{W_\varphi \mid \varphi \in V_\pi\}.$$

The group $G(\mathbb{A})$ acts in this space by right translation and the map

$$\varphi \mapsto W_\varphi \quad \text{intertwines} \quad V_\pi \xrightarrow{\sim} \mathcal{W}(\pi,\psi).$$

Note that since we can recover φ from W_φ through its Fourier expansion we are guaranteed that $W_\varphi \neq 0$ for all $\varphi \neq 0$. The space $\mathcal{W}(\pi,\psi)$ is called the *Whittaker model of* π.

The idea of a Whittaker model makes sense over a local field (and even a finite field). If we let k_v be a local field (a completion of our global field k) and let ψ_v be a non-trivial (continuous) additive character of k_v then as before ψ_v defines a character of the local translations $N(k_v)$. Let $\mathcal{W}(\psi_v)$ denote the full space of smooth functions $W : G(k_v) \to \mathbb{C}$ which satisfy $W(ng) = \psi_v(n)W(g)$ for all $n \in N(k_v)$. This is the space of smooth Whittaker functions on $G(k_v)$ and $G(k_v)$ acts on it by right translation.

If (π_v, V_{π_v}) is a smooth irreducible admissible representation of $G(k_v)$, then an intertwining

$$V_{\pi_v} \hookrightarrow \mathcal{W}(\psi_v) \quad \text{given by} \quad \xi_v \mapsto W_{\xi_v}$$

gives a Whittaker model $\mathcal{W}(\pi_v, \psi_v)$ of π_v.

For a representation (π_v, V_{π_v}) to have a Whittaker model it is necessary and sufficient for V_{π_v} to have a non-trivial (continuous) Whittaker functional, that is , a continuous functional $\Lambda_v : V_{\pi_v} \to \mathbb{C}$ satisfying

$$\Lambda_v(\pi_v(n)\xi_v) = \psi_v(n)\Lambda_v(\xi_v)$$

for all $n \in N(k_v)$ and $\xi_v \in V_{\pi_v}$. A model $\xi_v \mapsto W_{\xi_v}$ gives a functional by

$$\Lambda_v(\xi_v) = W_{\xi_v}(e)$$

and a functional Λ_v gives a model by setting

$$W_{\xi_v}(g) = \Lambda_v(\pi_v(g)\xi_v).$$

The fundamental result on local Whittaker models is due to Gelfand and Kazhdan ($v < \infty$) and Shalika ($v|\infty$).

Theorem 4.1 (Local Uniqueness). *Given (π_v, V_{π_v}) an irreducible admissible smooth representation of $G(k_v)$ the space of (continuous) Whittaker functionals is at most one dimensional, that is, and π_v has at most one Whittaker model.*

Remarks. (i) One proves this by showing that the space of Whittaker functions $\mathcal{W}(\psi_v)$ is multiplicity free as a representation of $G(k_v)$. Writing $\mathcal{W}(\psi_v) = Ind(\psi_v)^\infty$ one shows that the intertwining algebra of Bessel distributions B satisfying $B(n_1 g n_2) = \psi_v(n_1)B(g)\psi_v(n_2)$ is commutative by exhibiting an anti-involution of the algebra that stabilizes the individual distributions.
(ii) When $v|\infty$, if we worked simply with irreducible admissible representations of the Hecke algebra $\mathcal{H}_v$ then the space of (algebraic) Whittaker functionals on $(V_{\pi_v})_K$ would have dimension $n!$, but only one extends continuously to V_{π_v} with its (smooth moderate growth) Fréchet topology.
(iii) Simultaneously, one shows that if π_v has a Whittaker model, then so does its contragredient $\widetilde{\pi}_v$ and in fact

$$\mathcal{W}(\widetilde{\pi}_v, \psi_v^{-1}) = \{\widetilde{W}(g) = W(w_n\, {}^t g^{-1}) \mid W \in \mathcal{W}(\pi_v, \psi_v)\}$$

where w_n denotes the long Weyl element $w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$.

Definition 4.1. *A representation (π_v, V_{π_v}) having a Whittaker model is called generic.*

Of course, the same definition applies in the global situation. Note that for $G = GL_n$ this notion is independent of the choice of (nontrivial) ψ_v or ψ.

Now return to our smooth cuspidal representation (π, V_π), or in fact any irreducible admissible smooth representation of $G(\mathbb{A})$. If we factor π into its local components

$$\pi \simeq \otimes' \pi_v \quad \text{with} \quad V_\pi \simeq \otimes' V_{\pi_v}$$

then any Whittaker functional Λ on V_π determines a family of compatible Whittaker functionals Λ_v on the V_{π_v} by

$$\Lambda_v : V_{\pi_v} \hookrightarrow \otimes' V_{\pi_v} \xrightarrow{\sim} V_\pi \xrightarrow{\Lambda} \mathbb{C}$$

such that $\Lambda = \otimes \Lambda_v$. Similarly, any suitable family $\{\Lambda_v\}$ of Whittaker functionals on the V_{π_v}, where suitable means $\Lambda_v(\xi_v^\circ) = 1$ for our distinguished K_v-fixed vectors ξ_v° giving the restricted tensor product, determines a global Whittaker functional $\Lambda = \otimes \Lambda_v$ on $V_\pi = \otimes' V_{\pi_v}$.

The Local Uniqueness Theorem then has the following consequences.

Corollary 4.1.1 (Global Uniqueness). *If $\pi = \otimes' \pi_v$ is any irreducible admissible smooth representation of $G(\mathbb{A})$ then the space of Whittaker functionals of V_π is at most one dimensional, that is, π has a unique Whittaker model.*

If (π, V_π) is our cuspidal representation then we have seen that V_π has a global Whittaker functional given by

$$\Lambda(\varphi) = W_\varphi(e) = \int_{N(k)\backslash N(\mathbb{A})} \varphi(n)\psi^{-1}(n)\, dn.$$

Corollary 4.1.2. *If (π, V_π) is cuspidal with $\pi \simeq \otimes' \pi_v$ then π and each of its local components π_v are generic.*

A most important consequence for our purposes is:

Corollary 4.1.3 (Factorization of Whittaker Functions). *If (π, V_π) is a cuspidal representation with $\pi \simeq \otimes' \pi_v$ and $\varphi \in V_\pi$ such that under the isomorphism $V_\pi \simeq \otimes' V_{\pi_v}$ we have $\varphi \mapsto \otimes \xi_v$ (so φ is decomposable) then*

$$W_\varphi(g) = \prod_v W_{\xi_v}(g_v).$$

The proof is essentially the following simple computation:

$$W_\varphi(g) = \Lambda(\pi(g)\varphi) = (\otimes\Lambda_v)(\otimes\pi_v(g_v)\xi_v)$$
$$= \prod_v \Lambda_v(\pi_v(g_v)\xi_v) = \prod_v W_{\xi_v}(g_v).$$

Note once again that the cusp form $\varphi(g)$ itself does not factor. The $G(k)$-invariance mixes the various places together. Only W_φ factors for decomposable φ. If $f \in S_m(SL_2(\mathbb{Z}))$ is a classical Hecke eigen-form of weight m for $SL_2(\mathbb{Z})$, with its usual Fourier expansion $f(z) = \sum a_n e^{2\pi i n z}$, and $f \mapsto \varphi$ is our lifted automorphic form, then φ is decomposable and the Whittaker function W_φ factors. If we write $W_\varphi = W_\infty W_f$ then

$$W_\infty \begin{pmatrix} ny & \\ & 1 \end{pmatrix} = (ny)^{m/2} e^{-2\pi n y} \quad \text{and} \quad W_f \begin{pmatrix} n & \\ & 1 \end{pmatrix} = a_n.$$

4.3. **Multiplicity One for GL_n.** The uniqueness of the Whittaker model is the key to the following result.

Theorem 4.2 (Multiplicity One)**.** *Let (π, V_π) be a smooth irreducible admissible (unitary) representation of $GL_n(\mathbb{A})$. Then its multiplicity $m(\pi)$ in the space of cusp forms is at most one.*

This result was proven independently by Piatetski-Shapiro and Shalika, based on the Fourier expansion and the global uniqueness of Whittaker models. Suppose we have two realizations of π in the space of cusp forms:

$$V_\pi \hookrightarrow V_{\pi,i} \subset \mathcal{A}_0^\infty \quad \text{for} \quad i = 1, 2.$$

For $\xi \in V_\pi$ let φ_1 and φ_2 be the corresponding cusp forms. Then the maps

$$\xi \mapsto \varphi_i \mapsto W_{\varphi_i}(e) = \Lambda_i(\xi)$$

give two Whittaker functionals on V_π. By uniqueness, there exists $c \neq 0$ such that $\Lambda_1 = c\Lambda_2$. Then

$$W_{\varphi_1}(g) = \Lambda_1(\pi(g)\xi) = c\Lambda_2(\pi(g)\xi) = cW_{\varphi_2}(g)$$

so that

$$\varphi_1(g) = \sum_\gamma W_{\varphi_1}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right) = c \sum_\gamma W_{\varphi_2}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right) = c\varphi_2(g).$$

But then $V_{\pi,1} \cap V_{\pi,2} \neq \{0\}$. So by irreducibility $V_{\pi,1} = V_{\pi,2}$, that is, $m(\pi) = 1$.

4.4. **Strong Multiplicity One for GL_n.** Strong Multiplicity One was originally due to Piatetski-Shapiro. His proof, which we will sketch here, is a variant of the proof of Multiplicity One. We will give a second proof due to Jacquet and Shalika based on L-functions later.

Theorem 4.3 (Strong Multiplicity One for GL_n). *Let (π_1, V_{π_1}) and (π_2, V_{π_2}) be two cuspidal representations of $GL_n(\mathbb{A})$. Decompose them as $\pi_1 \simeq \otimes' \pi_{1,v}$ and $\pi_1 \simeq \otimes' \pi_{2,v}$. Suppose that there is a finite set of places S such that $\pi_{1,v} \simeq \pi_{2,v}$ for all $v \notin S$. Then $(\pi_1, V_{\pi_1}) = (\pi_2, V_{\pi_2})$.*

In place of the Whittaker model, Piatetski-Shapiro used a variant known as the *Kirillov model*. To define this, let

$$P = P_n = \left\{ \begin{pmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} = Stab\ ((0, \dots, 0, 1))$$

denote the *mirabolic subgroup* of GL_n. If we let (π_v, V_{π_v}) be an irreducible admissible generic representation of $G(k_v)$ with Whittaker model $\mathcal{W}(\pi_v, \psi_v)$ then we can consider the restrictions $W_v(p_v)$ of the functions $W_v \in \mathcal{W}(\pi_v, \psi_v)$ to $P_v = P(k_v)$. The first surprising fact is:

Theorem 4.4. *The map $W_v \mapsto W_v|_{P_v}$ is injective, that is, if $W_v \neq 0$ then $W_v(p_v) \not\equiv 0$.*

This is due to Bernstein and Zelevinsky if $v < \infty$ and Jacquet and Shalika if $v | \infty$.

Definition 4.2. *The (local) Kirillov model of a generic (π_v, V_{π_v}) is the space of functions on P_v defined by*

$$\mathcal{K}(\pi_v, \psi_v) = \{W_v(p_v) \mid W_v \in \mathcal{W}(\pi_v, \psi_v),\ p_v \in P_v\}.$$

A second surprising fact is that no matter what the generic representation (π_v, V_{π_v}) we begin with, the Kirillov models all have a common P_v sub-module, namely

$$\tau(\psi_v) = \begin{cases} ind_{N_v}^{P_v}(\psi_v) & v < \infty \\ \\ Ind_{N_v}^{P_v}(\psi_v)^\infty & v | \infty \end{cases}.$$

When $v | \infty$ Jacquet and Shalika established this only for π_v a local component of a cusp form. The result for $v < \infty$ is of course due to Bernstein and Zelevinsky. This is a canonical space of functions on P_v.

We may now sketch the proof of the Strong Multiplicity One theorem. Let π_1, π_2, and S be as in the statement of the theorem. As before, our goal is to produce a common non-zero cusp form $\varphi \in V_{\pi_1} \cap V_{\pi_2}$.

(i) Let $P' = P'_n = P_n Z_n$ be the $(n-1,1)$ parabolic subgroup of GL_n (here Z_n is still the center of GL_n). Then $P'(k)\backslash P'(\mathbb{A})$ is dense in $GL_n(k)\backslash GL_n(\mathbb{A})$. So it suffices to find $\varphi_i \in V_{\pi_i}$ such that $\varphi_1(p') = \varphi_2(p')$ for all $p' \in P'(\mathbb{A})$.

(ii) Utilizing the Fourier expansion as before, it suffices to find non-zero $W_i \in \mathcal{W}(\pi_i, \psi)$ such that $W_1(p') = W_2(p')$.

(iii) Since $\omega = \omega_{\pi_1} = \omega_{\pi_2}$ (by weak approximation) it suffices to find non-zero W_i such that $W_1(p) = W_2(p)$ for all $p \in P(\mathbb{A})$, that is, to find non-zero

$$W = W_1 = W_2 \in \mathcal{K}(\pi_1, \psi) \cap \mathcal{K}(\pi_2, \psi).$$

(iv) At $v \notin S$ we have $\pi_{1,v} \simeq \pi_{2,v}$ so that

$$\mathcal{K}(\pi_{1,v}\psi_v) = \mathcal{K}(\pi_{2,v}, \psi_v) \quad \text{for} \quad v \notin S.$$

At $v \in S$ we have

$$\tau(\psi_v) \subset \mathcal{K}(\pi_{1,v}\psi_v) \cap \mathcal{K}(\pi_{2,v}, \psi_v) \quad \text{for} \quad v \in S$$

which is a quite large intersection. So we may simply take any

$$W \in \prod_{v \in S} \tau(\psi_v) \prod_{v \notin S}{}' \mathcal{K}(\pi_{i,v}, \psi_v) \subset \mathcal{K}(\pi_1, \psi) \cap \mathcal{K}(\pi_2, \psi)$$

which is non-zero.

Now retrace the steps to obtain $\varphi \in V_{\pi_1} \cap V_{\pi_2}$ forcing $V_{\pi_1} = V_{\pi_2}$ as before.

Remark. In Piatetski-Shapiro's original proof, he had to require that the set S consisted of finite places, since the result of Jacquet and Shalika was not available at that time. Once it became available his proof worked for general finite set S as well.

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