3. Automorphic Representations

We have defined our spaces of automorphic forms. Now we turn to our tools. We will analyze $\mathcal{A}$, $\mathcal{A}^\infty$, or $L^2(\omega)$ as representation spaces for certain algebras or groups. Throughout we will let $G = GL_n$, although the results remain true for any reductive algebraic G, let k be a number field, and retain all notations from before.

3.1. (K-finite) automorphic representations.

As we have noted the space $\mathcal{A}$ of (K-finite) automorphic forms does not give a representation of $G(\mathbb{A})$. It will be a representation space for the global Hecke algebra $\mathcal{H}$.

3.1.1. *The Hecke algebra.* The global Hecke algebra $\mathcal{H}$ will be a restricted tensor product of local Hecke algebras: $\mathcal{H} = \otimes' \mathcal{H}_v$. $\mathcal{H}$ and each $\mathcal{H}_v$ will be idempotented algebras under convolution. So there will be a directed family of fundamental idempotents $\{\xi_i\}$ such that

$$\mathcal{H} = \varinjlim_i \xi_i * \mathcal{H} * \xi_i = \bigcup_i \xi_i * \mathcal{H} * \xi_i$$

and

$$\mathcal{H}_v = \varinjlim_i \xi_{i,v} * \mathcal{H}_v * \xi_{i,v} = \bigcup_i \xi_{i,v} * \mathcal{H} * \xi_{i,v}.$$

Neither $\mathcal{H}$ nor any $\mathcal{H}_v$ will have an identity, but for each ξ_i the subalgebra $\xi_i * \mathcal{H} * \xi_i$ will have ξ_i as an identity.

(i) If $v < \infty$ is an archimedean place of k then $\mathcal{H}_v = C_c^\infty(G(k_v))$ is the algebra of smooth (locally compact) compactly supported functions on $G_v = G(k_v)$. It is naturally an algebra under convolution. For each compact open subgroup $L_v \subset G_v$ there is a fundamental idempotent

$$\xi_{L_v} = \frac{1}{Vol(L_v)} \mathfrak{X}_{L_v}$$

where $\mathfrak{X}_{L_v}$ is the characteristic function of L_v. Then $\xi_{L_v} * \mathcal{H}_v * \xi_{L_v} = \mathcal{H}(G_v//L_v)$ is the algebra of L_v-bi-invariant compactly supported functions on G_v. In any representation of $\mathcal{H}_v$ the idempotent ξ_{L_v} will act as a projection onto the L_v-fixed vectors. We will let ξ_v° denote the fundamental idempotent associated to the maximal compact subgroup K_v. Note that if $k = \mathbb{Q}$, $G = GL_2$, and $L_p = K_p = GL_2(\mathbb{Z}_p)$ then $\xi_p^\circ * \mathcal{H}_p * \xi_p^\circ = \mathcal{H}(GL_2(\mathbb{Q}_p)//GL_2(\mathbb{Z}_p))$ is isomorphic to the complex algebra spanned by the classical Hecke operators $\langle T_{p^r} \rangle$.

(ii) If $v|\infty$ is an archimedean place of k then $\mathcal{H}_v$ is the convolution algebra of bi-K_v-finite distributions on G_v with support in K_v. Then $\mathcal{H}_v$ contains both

$$\mathcal{U}(\mathfrak{g}): \text{ distributions supported at the identity}$$

and

$$A(K_v): \text{ finite measures on } K_v$$

and in fact

$$\mathcal{H}_v = \mathcal{U}(\mathfrak{g}_v) \otimes_{\mathcal{U}(\mathfrak{k}_v)} A(K_v).$$

For each finite dimensional representation δ_v of K_v we have a fundamental idempotent

$$\xi_{\delta_v} = \frac{1}{\deg(\delta_v)} \Theta_{\delta_v}$$

where $\deg(\delta_v)$ is the degree and Θ_{δ_v} is the character of δ_v. In any representation δ_v should act as the projection onto the δ_v-isotypic component.

(iii) The global Hecke algebra $\mathcal{H}$ is then the restricted tensor product of the local algebras $\mathcal{H}_v$ with respect to the idempotents $\{\xi_v^\circ\}$ at the non-archimedean places., i.e.,

$$\mathcal{H} = \otimes_v' \mathcal{H}_v = \varinjlim_S \left((\otimes_{v \in S} \mathcal{H}_v) \otimes (\otimes_{v \notin S} \xi_v^\circ) \right)$$

as S runs over finite sets of places of k which contain all archimedean places $\mathcal{V}_\infty$. Let us write $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f$ where, as usual,

$$\mathcal{H}_\infty = \otimes_{v|\infty} \mathcal{H}_v \quad \text{and} \quad \mathcal{H}_f = \otimes'_{v<\infty} \mathcal{H}_v.$$

Then the fundamental idempotents in $\mathcal{H}$ are of the form $\xi = \xi_\infty \otimes \xi_f$ where

$$\xi_\infty = \xi_\delta = \otimes_{v|\infty} \xi_{\delta_v} \in \mathcal{H}_\infty$$

is associated to a finite dimensional representation $\delta = \otimes \delta_v$ of K_∞ and

$$\xi_f = \xi_L = \otimes_{v<\infty} \xi_{L_v} \in \mathcal{H}_f$$

is associated to a compact open subgroup $L = \prod L_v$ of G_f (so for almost all places $L_v = K_v$ and $\xi_{L_v} = \xi_v^\circ$).

3.1.2. *The representation on automorphic forms.* The space $\mathcal{A}$ of K-finite automorphic forms is naturally an $\mathcal{H}$-module by right convolution. For $\xi \in \mathcal{H}$ and $\varphi \in \mathcal{A}$ set

$$R(\xi)\varphi(g) = \varphi * \check{\xi}(g) = \int_{G(\mathbb{A})} \varphi(gh)\xi(h)\, dh$$

where $\check{\xi}(g) = \xi(g^{-1})$. Note that with this action the K-finiteness condition on $\varphi \in \mathcal{A}$ can now be stated as: there exists a fundamental idempotent $\xi = \xi_\infty \otimes \xi_f = \xi_\delta \otimes \xi_L$ such that $R(\xi)\varphi = \varphi$.

The representations that we will be most interested in will be *admissible* representations of $\mathcal{H}$.

Definition 3.1. *A representation (π_v, V_v) of a local Hecke algebra $\mathcal{H}_v$ is admissible if for every fundamental idempotent ξ_v we have*

$$\dim_{\mathbb{C}}(\pi_v(\xi_v)V_v) < \infty.$$

Similarly a representation (π, V) of the global Hecke algebra $\mathcal{H}$ is admissible if for every global fundamental idempotent $\xi \in \mathcal{H}$ the subspace $\pi(\xi)V$ is finite dimensional.

One consequence of admissibility, which we state in the global case, is that as a representation of K the space V decomposes into a direct sum of irreducibles with finite multiplicities:

$$V = \bigoplus_{\tau \in \hat{K}} m(\tau, V)V_\tau.$$

The reason for our interest in admissible representations is the following fundamental result of Harish-Chandra (probably first due to Jacquet and Langlands for GL_2).

Theorem 3.1. *Suppose $\varphi \in \mathcal{A}$. Then the $\mathcal{H}$-module generated by φ, namely*

$$V_\varphi = R(\mathcal{H})\varphi = \varphi * \mathcal{H} \subset \mathcal{A},$$

is an admissible $\mathcal{H}$-module.

This makes the following definition reasonable.

Definition 3.2. *An automorphic representation (π, V) of $\mathcal{H}$ is an irreducible (hence admissible) sub-quotient of $\mathcal{A}(G(k)\backslash G(\mathbb{A}))$.*

There is a canonical way to construct admissible representations of $\mathcal{H}$ abstractly using the restricted tensor product structure $\mathcal{H} = \otimes' \mathcal{H}_v$. Suppose we have a collection $\{(\pi_v, V_v)\}$ of admissible representations of the local Hecke algebras $\mathcal{H}_v$ such that for almost all finite places the representation V_v contains a (fixed) K_v-invariant vector, say u_v°. Then we can define the restricted tensor product of these representations with respect to the $\{u_v^\circ\}$ in the (by now) usual manner:

$$V = \otimes_v' V_v = \varinjlim_S \left((\otimes_{v \in S} V_v) \otimes (\otimes_{v \notin S} u_v^\circ) \right).$$

Note that since u_v° is K_v-fixed, then $\pi_v(\xi_v^\circ) u_v^\circ = u_v^\circ$ so this space does carry a natural representation of $\mathcal{H}$, coming from its restricted tensor product decomposition, which we will denote by $\pi = \otimes' \pi_v$. We leave it as an exercise to verify that if each of the (π_v, V_v) is admissible then so is (π, V) and if each (π_v, V_v) is irreducible, then so is (π, V).

An important fact for us, which is a purely algebraic fact about $\mathcal{H}$-modules, is the converse to this construction.

Theorem 3.2 (Decomposition Theorem)**.** *If (π, V) is an irreducible admissible representation of $\mathcal{H}$ then for each place v of k there exists an irreducible admissible representation (π_v, V_v) of $\mathcal{H}_v$, having a K_v-fixed vector for almost all v, such that $\pi = \otimes' \pi_v$.*

Therefore in the context of automorphic representations of $\mathcal{H}$ we have the following corollary.

Corollary 3.2.1. *If (π, V) is an automorphic representation, then π decomposes into a restricted tensor product of local irreducible admissible representations: $\pi = \otimes' \pi_v$.*

Note that the decomposition given in this corollary is an abstract decomposition. It does not give a factorization of automorphic forms into a product of functions on the local groups $G(k_v)$.

3.2. **Smooth automorphic representations.** Now things are more straight forward on the one hand, since $G(\mathbb{A})$ acts in $\mathcal{A}^\infty(G(k)\backslash G(\mathbb{A}))$ by right translation. However the representation theory is now a bit more complicated. More precisely, for every compact open subgroup $L \subset K_f$ the space of L-invariant functions $(\mathcal{A}^\infty)^L$ in $\mathcal{A}^\infty$, namely

$$(\mathcal{A}^\infty)^L = \{\varphi \in \mathcal{A}^\infty \mid \varphi(g\ell) = \varphi(g) \quad \text{for} \quad \ell \in L\},$$

is a representation for G_∞. The spaces $(\mathcal{A}^\infty)^L$ all carry compatible limits of smooth Fréchet topologies coming from the uniform moderate growth semi-norms on $\mathcal{A}^\infty$ and the representation of G_∞ on these spaces are limits of smooth Fréchet representation of moderate growth. Then as a topological representation

$$\mathcal{A}^\infty = \bigcup_L (\mathcal{A}^\infty)^L = \varinjlim_L (\mathcal{A}^\infty)^L$$

also carries a limit-Fréchet topology. Without going into details on such representations, let us state the results we will need analogous to those for representations of $\mathcal{H}$.

Theorem 3.3 (Wallach). *If $\varphi \in \mathcal{A}^\infty$ is a smooth automorphic form then the (closed) sub-representation generated by φ, namely*

$$V_\varphi = \overline{R(G(\mathbb{A}))\varphi} \subset \mathcal{A}^\infty,$$

is admissible *in the sense that its (dense) subspace of K-finite vectors $(V_\varphi)_K$ is admissible as an $\mathcal{H}$-module.*

Then we can make the following definition.

Definition 3.3. *A smooth automorphic representation (π, V) of $G(\mathbb{A})$ is a (closed) irreducible sub-quotient of $\mathcal{A}^\infty(G(k)\backslash G(\mathbb{A}))$.*

Note that the smooth automorphic representations are automatically admissible in the above sense. We still have a version of the Decomposition Theorem, which we state as follows.

Theorem 3.4 (Decomposition Theorem). *If (π, V) is a smooth automorphic representation of $G(\mathbb{A})$ then there exist irreducible admissible smooth representations (π_v, V_v) of $G(k_v)$, which are smooth Fréchet representations of moderate growth if $v|\infty$, such that $\pi = \pi_\infty \otimes \pi_f$ where*

$$\pi_\infty = \widehat{\otimes}_{v|\infty}\pi_v$$

is the topological tensor product of smooth Fréchet representations and

$$\pi_f = \otimes'_{v<\infty}\pi_v$$

is the restricted tensor product of smooth representations of the $G(k_v)$. Moreover, if (π_K, V_K) is the associated irreducible $\mathcal{H}$-module of K-finite vectors in V then in the decomposition $\pi_K = \otimes'(\pi_K)_v$ we have $\pi_v = (\pi_K)_v$ for $v < \infty$ while for $v|\infty$ we have $(\pi_v)_K = (\pi_K)_v$ and $\pi_v = \widehat{(\pi_K)_v}$ is the Casselman-Wallach canonical completion of the $\mathcal{H}_v$-module $(\pi_K)_v$.

Even though the theory of smooth automorphic representations is topological, according to Wallach it is also quite algebraic. These representations will be algebraically irreducible as representations of the global *Schwartz algebra* $\mathcal{S} = \mathcal{S}(G(\mathbb{A}))$. This is a restricted tensor product of the local Schwatrz algebras $\mathcal{S}_v = \mathcal{S}(G(k_v))$. For archimedean places $v|\infty$ then $\mathcal{S}_v$ is the usual space of smooth (infinitely differentiable) rapidly decreasing functions on $G(k_v)$. At the non-archmiedean places rapidly decreasing is interpreted as having compact support, so $\mathcal{S}_v$ is the space of smooth (locally constant) compactly supported supported functions on $G(k_v)$, that is, $\mathcal{S}_v = \mathcal{H}_v$. Then $\mathcal{S} = \mathcal{S}_\infty \otimes \mathcal{S}_f$ where now

$$\mathcal{S}_\infty = \mathcal{S}(G_\infty) = \widehat{\otimes}_{v|\infty}\mathcal{S}_v \quad \text{and} \quad \mathcal{S}_f = \otimes'_{v<\infty}\mathcal{S}_v = \mathcal{H}_f.$$

3.3. **L^2-automorphic representations.** If we now fix a unitary central character $\omega : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ and consider the associated space of L^2-automorphic forms $L^2(G(k)\backslash G(\mathbb{A});\omega)$ then this space is a Hilbert space and affords a unitary representation representation of $G(\mathbb{A})$ acting by right translation. In some sense this is the easiest situation to be in.

Theorem 3.5 (Harish-Chandra)**.** *If $\varphi \in L^2(\omega)$ then*

$$V_\varphi = \overline{R(G(\mathbb{A}))\varphi} \subset L^2(\omega)$$

is an admissible sub-representation in the sense that the (dense) subspace $(V_\varphi)_K$ of K-finite vectors is admissible as as $\mathcal{H}$-module.

Definition 3.4. *An L^2-automorphic representation (π, V) is an irreducible constituent in the L^2-decomposition of some $L^2(\omega)$.*

In the context of L^2-automorphic representations, the Decomposition Theorem predates the algebraic one and is due to Gelfand and Piatetski-Shapiro.

Theorem 3.6. *If (π, V) is an L^2-automorphic representation then there exist irreducible unitary representations (π_v, V_v) of $G(k_v)$ such that $\pi = \widehat{\otimes}'\pi_v$ is a restricted Hilbert tensor product of local representations.*

3.4. **Cuspidal representations.** Since the cuspidality condition is defined by the vanishing of a left unipotent integration

$$\int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\, du = 0,$$

which is a closed condition, and our actions of $\mathcal{H}$ or $G(\mathbb{A})$ on the spaces of automorphic forms are by right convolution or right translations we see that the spaces of cusp forms $\mathcal{A}_0$, $\mathcal{A}_0^\infty$, or $L^2_0(\omega)$ are all (closed) sub-representations of the relevant spaces of automorphic forms.

A fundamental result of the space of L^2-cusp forms is the following result of Gelfand and Piatetski-Shapiro.

Theorem 3.7. *The space $L_0^2(\omega)$ of L^2-cusp forms decomposes into a discrete Hilbert direct sum with finite multiplicities of irreducible unitary sub-representations:*

$$L_0^2(\omega) = \oplus m(\pi) V_\pi \quad \textit{with} \quad m(\pi) < \infty.$$

We can then make the following definition.

Definition 3.5. *The irreducible constituents (π, V_π) of the various $L_0^2(\omega)$ are the L^2-cuspidal representations.*

Recall that for a fixed unitary central character ω we have, as a consequence of the rapid decrease of cusp forms, the inclusions

$$\mathcal{A}_0(\omega) \subset \mathcal{A}_0^\infty(\omega) \subset L_0^2(\omega)$$

and in fact upon passing to smooth vectors and then K-finite vectors we have

$$\mathcal{A}_0^\infty(\omega) = L_0^2(\omega)^\infty \quad \text{and} \quad \mathcal{A}_0(\omega) = \mathcal{A}_0^\infty(\omega)_K = L_0^2(\omega)_K$$

so we can deduce the decompositions

$$\mathcal{A}_0^\infty(\omega) = \oplus m(\pi) V_\pi^\infty \quad \text{and} \quad \mathcal{A}_0(\omega) = \oplus m(\pi)(V_\pi)_K.$$

Definition 3.6. *The irreducible constituents of $\mathcal{A}_0(\omega)$ are the unitary (K-finite) cuspidal representations of $G(\mathbb{A})$ and the irreducible constituents of $\mathcal{A}^\infty(\omega)$ are the unitary smooth cuspidal representations of $G(\mathbb{A})$.*

Note that if (π, V_π) is a cuspidal representation (in any context) then the elements of V_π are indeed cusp forms, that is, $V_\pi \subset \mathcal{A}_0$ as a subspace not a more general sub-quotient.

In general any irreducible subrepresentation of $\mathcal{A}_0$ or $\mathcal{A}_0^\infty$ will be called a cuspidal representation. Due to the rapid decrease of cusp forms, any cuspidal representation (π, V_π) will be an unramified twist of a unitary cuspidal representation, that is, if we define for any character $\chi : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ the twisted representation $\pi \otimes \chi$ as the representation by right translation on the space $V \otimes \chi = \{\varphi(g)\chi(\det g) \mid \varphi \in V_\pi\}$, then one can always find an unramified character χ such that $\pi \otimes \chi$ is a unitary cuspidal representation as above. Some choose to call the non-unitary cuspidal representations quasi-cuspidal.

3.5. **Connections with classical forms.** Suppose we return to a classical cusp form f for $SL_2(\mathbb{Z})$ of weight m. If we follow our passage $f \mapsto \varphi \mapsto (\pi_\varphi, V_\varphi)$ then (π_φ, V_φ) is an admissible subspace of the space of cuspidal automorphic forms. It need not be irreducible. However, if in addition f is a simultaneous eigen-function for all the classical Hecke operators, then (π_φ, V_φ) is irreducible and hence a cuspidal representation. Then the Decomposition Theorem lets us decompose π_φ as $\pi_\varphi = \pi_\infty \otimes (\otimes' \pi_p)$. In this decomposition

(i) π_∞ is completely determined by the weight m of f
(ii) π_p is completely determined by the Hecke eigen-value $\lambda(p)$ of T_p acting on f.

In fact, as we shall see, the Decomposition Theorem for π_φ is equivalent to the Euler product factorization for the completed L-function $\Lambda(s, f)$.

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