11.5. **Global functoriality.** Now let us take $k$ a global field of characteristic 0, that is, a number field and let $G$ be a connected reductive group defined and split over $k$. As before, we let $r : {}^LG \to GL_n(\mathbb{C}) = {}^LGL_n$ be an $L$-homomorphism. Then there is also a global Principle of Functoriality, namely:

**Principle of Functoriality:** *Associated to the $L$-homomorphism $r : {}^LG \to {}^LGL_n$ there should be associated a natural lift or transfer of automorphic representations of $G(\mathbb{A})$ to automorphic representations of $GL_n(\mathbb{A})$ .*

We can give a precise formulation of this through local Langlands functoriality and a local-global principle. Let $\pi = \otimes' \pi_v$ be an irreducible automorphic representation of $G(\mathbb{A})$. Then there is a finite set $S$ of finite places such that for all $v \notin S$ we have that either $v$ is archimedean or $\pi_v$ is unramified. In either case, we understand the local Langlands conjecture for $\pi_v$ and hence we have a local functorial lift $\Pi_v$ as a representation of $GL_n(k_v)$.

**Definition 11.1.** *Let $\pi = \otimes' \pi_v$ be an automorphic representation of $G(\mathbb{A})$. An automorphic representation $\Pi = \otimes' \Pi_v$ of $GL_n(\mathbb{A})$ will be called a* functorial lift or transfer *of $\pi$ if there is a finite set of places $S$ such that $\Pi_v$ is the local Langlands lift of $\pi_v$ for all $v \notin S$.*

Then Langlands' Principle of Functoriality predicts that every automorphic representation $\pi$ of $G(\mathbb{A})$ does indeed have a functorial lift to $GL_n(\mathbb{A})$. Note that $\Pi$ being a functorial lift of $\pi$ entails an equality of partial $L$-functions $L^S(s, \pi, r) = L^S(s, \Pi)$ as well as for $\varepsilon$-factors and twisted versions. (A one point we called this a weak lift. But the terminology of functorial lift (without any prejudicial adjective) is consistent with the recent formulations of functoriality due to Arthur and Langlands himself.)

11.6. **Functoriality and the Converse Theorem.** It should be clear how to approach the problem of global functoriality via the Converse Theorem. We begin with a cuspidal automorphic representation $\pi = \otimes' \pi_v$ of $G(\mathbb{A})$. There are three basic steps:

1. *Construction of a candidate lift.* If we know the local Langlands conjecture for all $\pi_v$ then we simply take for $\Pi_v$ the local Langlands

lift of $\pi_v$. Note that these local lifts will satisfy

$$L(s,\pi_v\times\pi_v',r\otimes\iota)=L(s,\Pi_v\times\pi_v')$$
$$\varepsilon(s,\pi_v\times\pi_v',r\otimes\iota,\psi_v)=\varepsilon(s,\Pi_v\times\pi_v',\psi_v)$$

for all irreducible admissible generic representations $\pi_v'$ of $GL_r(k_v)$. Then we take $\Pi=\otimes'\Pi_v$ to be our candidate lift. We then have

$$L(s,\pi\times\pi',r\otimes\iota)=L(s,\Pi\times\pi')$$
$$\varepsilon(s,\pi\times\pi',r\otimes\iota)=\varepsilon(s,\Pi\times\pi')$$

for all cuspidal $\pi'$ of $GL_r(\mathbb{A})$.

In practice, there will be a finite set of places $S$ where do not know the local Langlands conjecture for $\pi_v$ and we will have to deal with this.

2. *Analytic properties of $L$-functions.* By the equality of $L$- and $\varepsilon$-factors above, to show that $L(s,\Pi\times\pi')$ is nice for $\pi'$ in a suitable twisting set $\mathcal{T}$ it suffices to know this for $L(s,\pi\times\pi',r\otimes\iota)$. But this is what Kim has been lecturing on all semester.

In practice we do not expect $L(s,\pi\times\pi')$ to be entire always, since we do expect some cuspidal representations $\pi$ of $G(\mathbb{A})$ to lift to non-cuspidal representations $\Pi$ of $GL_n(\mathbb{A})$. This will also have to be dealt with.

3. *Apply the Converse Theorem.* Once we know that $L(s,\Pi\times\pi')$ is nice for a suitable twisting set $\mathcal{T}$, then we can apply the appropriate Converse Theorem to conclude that a functorial lift exists.

We have left two problems unresolved: (i) the lack of the local Langlands conjecture at the $v\in S$, and (ii) the fact that some $L(s,\pi\times\pi')$ could have poles. We are able to finesse both of these using an appropriately chosen idele class character $\eta$ and the Useful Variant of our Converse Theorems. We will explain these in the next lecture when we deal with the functoriality for the classical groups.

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## 12. FUNCTORIALITY FOR THE CLASSICAL GROUPS

We again take $k$ to be a number field. In this Lecture, this is currently a necessary restriction. We let $\mathbb{A}$ denote its ring of adeles and fix a non-trivial character $\psi$ of $k\backslash\mathbb{A}$.

12.1. **The result.** We take $G = G_n$ to be a split classical group of rank $n$ defined over $k$. More specifically, we consider the following cases.

(a) $G_n = SO_{2n+1}$ or $SO_{2n}$, the special orthogonal group over $k$ with respect to the symmetric bilinear form represented by

$$\Phi_m = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \quad \text{with } m = 2n+1,\ 2n.$$

(b) $G_n = Sp_{2n}$ the symplectic group with respect to the alternating form represented by

$$J_{2n} = \begin{pmatrix} & \Phi_n \\ -\Phi_n & \end{pmatrix}.$$

In each case, there is a standard embedding $r : {}^LG \hookrightarrow GL_N(\mathbb{C}) = {}^LGL_N$ for an appropriate $N$ as given in the following table.

| $G_n$ | $r : {}^LG_n \hookrightarrow {}^LGL_N$ | $GL_N$ |
|---|---|---|
| $SO_{2n+1}$ | $Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ | $GL_{2n}$ |
| $SO_{2n}$ | $SO_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ | $GL_{2n}$ |
| $Sp_{2n}$ | $SO_{2n+1}(\mathbb{C}) \hookrightarrow GL_{2n+1}(\mathbb{C})$ | $GL_{2n+1}$ |

Let $\pi = \otimes' \pi_v$ be a globally generic cuspidal representation of $G_n(\mathbb{A})$. [Recall that if $B = TU$ is the standard (upper triangular) Borel subgroup of $G(\mathbb{A})$ and we extend our additive character to one of $U(k)\backslash U(\mathbb{A})$ in the standard way then $\pi$ is globally generic if for $\varphi \in V_\pi$ we have

$$\int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\psi^{-1}(u)\ du \not\equiv 0.]$$

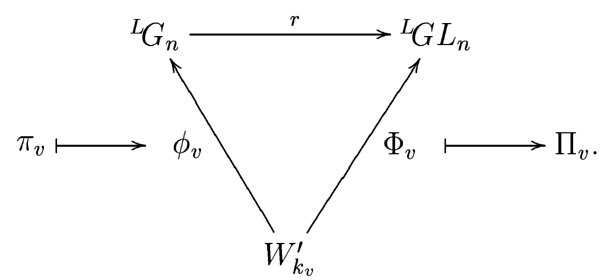
Our result is then the following.

**Theorem 12.1.** *Let $\pi$ be a globally generic cuspidal representation of $G_n(\mathbb{A})$. Then $\pi$ has a functorial lift $\Pi$ to $GL_N(\mathbb{A})$.*

Our proof will be by the Converse Theorem. We will follow the three steps given above.

Let $S$ be a non-empty set of finite places such that $\pi_v$ is unramified for all finite $v \notin S$.

12.2. **Construction of a candidate lift.** (i) If $v \notin S$, then either $v|\infty$ or $v < \infty$ and $\pi_v$ is unramified. In either case we have the local Langlands parameterization for $\pi_v$ and hence a local functorial lift $\Pi_v$ as an irreducible admissible representation of $GL_N(k_v)$.




As suggested by the formalism, one can show the following.

**Proposition 12.1.** *Let $\Pi_v$ be the local functorial lift of $\pi_v$. Let $\pi'_v$ be an irreducible admissible generic representation of $GL_d(k_v)$ with $1 \leq d \leq N-1$. Then*

$$L(s, \pi_v \times \pi'_v) = L(s, \Pi_v \times \pi'_v)$$
$$\varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \varepsilon(s, \Pi_v \times \pi'_v, \psi_v)$$

For simplicity, since $r$ is the standard embedding of the $L$-groups, we have dropped it from our notation and written

$$L(s, \pi_v \times \pi'_v) = L(s, \pi_v \times \pi'_v, r \otimes \iota),$$

etc..

(ii) If $v \in S$ then we may not have the local Langlands parameterization of $\pi_v$. We replace this knowledge with the following two local results.

**Proposition 12.2** (Multiplicativity of $\gamma$). *If $\pi_v$ is an irreducible admissible generic representation of $G_n(k_v)$ and $\pi'_v$ is an irreducible admissible generic representation of $GL_d(k_v)$ of the form*

$$\pi'_v \simeq \mathrm{Ind}_{Q(k_v)}^{GL_d(k_v)}(\pi'_{1,v} \otimes \pi'_{2,v})$$

*then*

$$\gamma(s, \pi_v \times \pi'_v, \psi_v) = \gamma(s, \pi_v \times \pi'_{1,v}, \psi_v)\gamma(s, \pi_v \times \pi'_{2,v}, \psi_v).$$

In this case, one also has a divisibility among the $L$-functions

$$L(s, \pi_v \times \pi'_v)^{-1} \Big| \, [L(s, \pi_v \times \pi'_{1,v})L(s, \pi_v \times \pi'_{2,v})]^{-1}.$$

There is a similar multiplicativity in the first variable, that is, when the representation $\pi_v$ of $G_n(k_v)$ is induced.

**Proposition 12.3** (Stability of $\gamma$). *Let $\pi_{1,v}$ and $\pi_{2,v}$ be two irreducible admissible smooth generic representations of $G_n(k_v)$. Then for every sufficiently highly ramified character $\eta_v$ of $k_v^\times$ we have*

$$\gamma(s, \pi_{1,v} \times \eta_v, \psi_v) = \gamma(s, \pi_{2,v} \times \eta_v, \psi_v).$$

In this situation, one also has that the $L$-functions stabilize

$$L(s, \pi_{1,v} \times \eta_v) = L(s, \pi_{2,v} \times \eta_v) \equiv 1$$

so that the $\varepsilon(s, \pi_{i,v} \times \eta_v, \psi_v)$ stabilize as well.

Recall from Lecture 6 that we had analogous statements for $GL_n(k_v)$. Moreover, as noted there, by using the multiplicativity in the $G_n$-variable one can compute the stable form of the $\gamma$-factor in terms of abelian $\gamma$-factors. Comparing these stable forms for $G_n(k_v)$ with those for $GL_N(k_v)$ one finds:

**Proposition 12.4** (Comparison of stable forms). *Let $\pi_v$ be an irreducible admissible generic representation of $G_n(k_v)$. Let $\Pi_v$ be an irreducible admissible representation of $GL_N(k_v)$ having trivial central character. Then for every sufficiently ramified character $\eta_v$ of $GL_1(k_v)$ we have*

$$\gamma(s, \pi_v \times \eta_v, \psi_v) = \gamma(s, \Pi_v \times \eta_v, \psi_v).$$

Of course since both $L$-functions stabilize to 1, this gives the equality of the stable $L$- and $\varepsilon$-factors.

So at the places $v \in S$ we can now take as the local component $\Pi_v$ of our candidate lift any irreducible admissible representation of $GL_N(k_v)$ with $\omega_{\Pi_v} \equiv 1$. With this choice of $\Pi_v$ we have the following result.

**Proposition 12.5.** *Let $\pi'_v$ be an irreducible admissible generic representation of $GL_d(k_v)$ of the form $\pi'_v = \pi'_{0,v} \otimes \eta_v$ with $\pi'_{0,v}$ unramified and $\eta_v$ chosen as above. Then we have*

$$L(s, \pi_v \times \pi'_v) = L(s, \Pi_v \times \pi'_v)$$
$$\varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \varepsilon(s, \Pi_v \times \pi'_v, \psi_v)$$

To see this on the level of $\gamma$-factors, write $\pi'_{0,v} = \mathrm{Ind}(|\ |_v^{s_1} \otimes \cdots \otimes |\ |_v^{s_d})$. Then $\pi'_v = \mathrm{Ind}(|\ |_v^{s_1}\eta_v \otimes \cdots \otimes |\ |_v^{s_d}\eta_v)$ and we have

$$\begin{aligned}\gamma(s, \pi_v \times \pi'_v, \psi_v) &= \prod_{i=1}^{d} \gamma(s + s_i, \pi_v \times \eta_v, \psi_v) && \text{(multiplicativity)}\\ &= \prod_{i=1}^{d} \gamma(s + s_i, \Pi_v \times \eta_v, \psi_v) && \text{(stability)}\\ &= \gamma(s, \Pi_v \times \pi'_v, \psi_v) && \text{(multiplicativity)}\end{aligned}$$

Return to our generic cuspidal representation $\pi = \otimes' \pi_v$ of $G_n(\mathbb{A})$. For each $\pi_v$ we have attached a local representation $\Pi_v$ of $GL_N(k_v)$, which is the local functorial lift for those $v \notin S$. Then $\Pi = \otimes' \Pi_v$ is an irreducible admissible representation of $GL_N(\mathbb{A})$. This is our candidate lift. Combining our local results, we have:

**Proposition 12.6.** *Let $\pi$ and $\Pi$ be as above. Then there exists an idele class character $\eta : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ such that for all $\pi' \in \mathcal{T}^S(N-1) \otimes \eta$ we have*

$$L(s, \pi \times \pi') = L(s, \Pi \times \pi')$$
$$\varepsilon(s, \pi \times \pi') = \varepsilon(s, \Pi \times \pi')$$

12.3. **Analytic properties of $L$-functions.** The analytic properties of the $L(s, \pi \times \pi')$ are controlled through the Fourier coefficients of Eisenstein series as in Kim's lectures. We summarize the results from there that we need in the following result.

**Proposition 12.7.** *Let $\pi$ be a globally generic cuspidal representation of $G_n(\mathbb{A})$. Let $S$ be a non-empty set of finite places and let $\eta : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be an idele class character such that at one place $v_0 \in S$ we have that both $\eta_{v_0}$ and $\eta_{v_0}^2$ are ramified. Then $L(s, \pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}^S(N-1) \otimes \eta$, that is,*

(i) *$L(s, \pi \times \pi')$ and $L(s, \widetilde{\pi} \times \widetilde{\pi}')$ are entire functions of $s$;*
(ii) *these functions are bounded in vertical strips;*

(iii) *we have the standard functional equation*

$$L(s,\pi\times\pi')=\varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

Recall that $\eta$ is necessary only to ensure that the $L(s,\pi\times\pi')$ are all entire. This resolves our global problem that the lift $\Pi$ of $\pi$ need not be cuspidal so that $L(s,\pi\times\pi')$ might have poles if some restriction is not placed on the $\pi'$.

It is in the use of the Eisenstein series to control the $L$-functions that $k$ is required to be a number field. In reality this should not matter, but at present this method of controlling the $L$-functions is only worked out in characteristic zero, that is, the number field case.

12.4. **Apply the Converse Theorem.** Take $\pi=\otimes'\pi_v$ to be our globally generic cuspidal representation of $G_n(\mathbb{A})$. Let $S$ be a non-empty set of finite places such that $\pi_v$ is unramified for all finite places $v\notin S$. Construct the candidate lift $\Pi=\otimes'\Pi_v$ as above.

For an appropriate choice if idele class character $\eta: k^\times\backslash\mathbb{A}^\times\to\mathbb{C}^\times$, chosen to satisfy both our local requirements of Proposition 12.6 and our global requirement of Proposition 12.7, we know that for all $\pi'\in\mathcal{T}^S(N-1)\otimes\eta$ both

$$L(s,\pi\times\pi')=L(s,\Pi\times\pi')$$
$$\varepsilon(s,\pi\times\pi')=\varepsilon(s,\Pi\times\pi')$$

and

$$L(s,\pi\times\pi'),\ \text{and hecne } L(s,\Pi\times\pi'),\ \text{is nice.}$$

Now applying the useful variant of our Converse Theorem there exists an automorphic representation $\Pi'$ of $GL_N(\mathbb{A})$ such that for all $v\notin S$ we have

$$\Pi'_v\simeq\Pi_v=\text{ the local functional lift of }\ \pi_v.$$

Then $\Pi'$ is our functorial lift of $\pi$.

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