10. Converse Theorems

Once again, we take k to be a global field, which we have taken to be a number field – but that is irrelevant. Then $\mathbb{A}$ is its ring of adeles and we take $\psi : k\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$ a non-trivial continuous additive character.

10.1. Converse Theorems for GL_n.

For automorphic representations of $GL_n(\mathbb{A})$ the "Converse Theorem", i.e., the converse to the theory of global L-functions developed in the last lecture, has a slightly different flavor from the classical ones. It addresses the following question.

Let us take $\pi \simeq \otimes' \pi_v$ to be an arbitrary (i.e., not necessarily automorphic) irreducible admissible smooth representation of $GL_n(\mathbb{A})$.

Question: *How can we tell if the local pieces π_v of π are "coherent enough" that we have an embedding*

$$V_\pi \hookrightarrow \mathcal{A}_0^\infty(GL_n(k)\backslash GL_n(\mathbb{A}))?$$

Our Converse Theorems gives an analytic answer to this question in terms of L-functions. From our local theory of L-functions, to each local component π_v we have attached a local L-factor $L(s, \pi_v)$ and a local ε-factor $\varepsilon(s, \pi_v, \psi_v)$. Thus we can (at least formally) form the Euler products

$$L(s,\pi) = \prod_v L(s,\pi_v) \quad \text{and} \quad \varepsilon(s,\pi,\psi) = \prod_v \varepsilon(s,\pi_v,\psi_v).$$

Then $L(s, \pi)$ is a formal Euler product of degree n and our question can be rephrased as:

Question: *Is the Dirichlet series defined by this formal Euler product modular?*

This is closer to the classical Converse Theorems.

To begin we must make some mild coherence and modularity assumptions, namely that

(i) the Euler product for $L(s, \pi)$ is absolutely convergent in some right half plane $Re(s) >> 0$;
(ii) the central character ω_π of π is an automorphic form on $GL_1(\mathbb{A})$, that is, an idele class character of $k^\times\backslash\mathbb{A}^\times$.

Note that one can show that (ii) implies that $\varepsilon(s,\pi,\psi)=\varepsilon(s,\pi)$ is independent of ψ.

Under these conditions, if $\pi'\simeq\pi'_v$ is any *cuspidal* (hence automorphic) representation of $GL_m(\mathbb{A})$ with $1\le m\le n-1$ then we can similarly form

$$L(s,\pi\times\pi')=\prod_v L(s,\pi_v\times\pi'_v) \text{ and } \varepsilon(s,\pi\times\pi',\psi)=\prod_v \varepsilon(s,\pi_v\times\pi'_v,\psi_v)$$

and still have that

- both the Euler products for $L(s,\pi\times\pi')$ and $L(s,\widetilde{\pi}\times\widetilde{\pi}')$ converge absolutely for $Re(s)>>0$; and that
- $\varepsilon(s,\pi\times\pi',\psi)=\varepsilon(s,\pi\times\pi')$ is independent of ψ.

We say that $L(s,\pi\times\pi')$ is *nice* if it behaves as it would if π were cuspidal, i.e.,

(i) $L(s,\pi\times\pi')$ and $L(s,\widetilde{\pi}\times\widetilde{\pi}')$ extend to entire functions of s;
(ii) these extensions are bounded in vertical strips;
(iii) they satisfy the functional equation

$$L(s,\pi\times\pi')=\varepsilon(s,\pi\times\pi')L(s,\widetilde{\pi}\times\widetilde{\pi}').$$

Our Converse Theorems, like Weil's, will involve these twists. To that end, for any m with $1\le m\le n-1$ let us set

$$\mathcal{T}(m)=\coprod_{d=1}^{m}\{\pi' \text{ cuspidal}, \ \ V_{\pi'}\subset\mathcal{A}_0^\infty(GL_d(k)\backslash GL_d(\mathbb{A}))\}$$

and for any finite set S of finite places we set

$$\mathcal{T}^S(m)=\{\pi'\in\mathcal{T}(m)\mid \pi'_v \text{ is unramified for all } \ v\in S\}.$$

The basic Converse Theorem, the analogue of those of Hecke and Weil, is the following result.

Theorem 10.1. *Let π be as above, an irreducible admissible smooth representation of $GL_n(\mathbb{A})$ having automorphic central character and such that $L(s,\pi)$ converges for $Re(s)>>0$. Let S be a finite set of finite places. Suppose that $L(s,\pi\times\pi')$ is nice for all $\pi'\in\mathcal{T}^S(n-1)$. Then*

(i) *if $S=\emptyset$ then π is automorphic and cuspidal;*

(ii) *if* $S \neq \emptyset$ *then* π *is quasi-automorphic in the sense that there exists an automorphic representation* π_1 *such that* $\pi_{1,v} \simeq \pi_v$ *for all* $v \notin S$.

A stronger result, but somewhat harder to prove, is the following.

Theorem 10.2. *Let* $n \geq 3$ *and let* π *be as above, an irreducible admissible smooth representation of* $GL_n(\mathbb{A})$ *having automorphic central character and such that* $L(s, \pi)$ *converges for* $Re(s) >> 0$. *Let* S *be a finite set of finite places. Suppose that* $L(s, \pi \times \pi')$ *is nice for all* $\pi' \in \mathcal{T}^S(n-2)$. *Then*

(i) *if* $S = \emptyset$ *then* π *is automorphic and cuspidal;*
(ii) *if* $S \neq \emptyset$ *then* π *is quasi-automorphic in the sense that there exists an automorphic representation* π_1 *such that* $\pi_{1,v} \simeq \pi_v$ *for all* $v \notin S$.

I would like to sketch the proof of Theorem 10.1. For simplicity let us assume that in addition (π, V_π) is generic. (We have discussed how to get around this in practice.)

10.2. **Inverting the integral representation.** Take $\pi \simeq \otimes' \pi_v$ as in the statement of Theorem 10.1. For this section we assume that $L(s, \pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(n-1)$ and see what this leads to when we invert our integral representation.

We first need to produce some functions on $GL_n(\mathbb{A})$. Since we have assumed (π, V_π) is generic we can do this via the Whittaker model. If $\xi \in V_\pi$ is such that under the decomposition $V_\pi \simeq \otimes' V_{\pi_v}$ we have $\xi \simeq \otimes \xi_v$ then to each ξ_v we have associated a Whittaker function $W_{\xi_v} \in \mathcal{W}(\pi_v, \psi_v)$ and hence

$$W_\xi(g) = \prod_v W_{\xi_v}(g_v) \in \mathcal{W}(\pi, \psi)$$

is a smooth function on $N_n(k) \backslash GL_n(\mathbb{A})$.

We could try to embed V_π into $\mathcal{A}_0^\infty$ by averaging W_ξ over $GL_n(k)$, but this would not converge. However the standard estimates on Whittaker functions do let us average over the rational points of the mirabolic

$$P = \mathrm{Stab}_{GL_n}((0,\dots,0,1)) = \left\{ p = \begin{pmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\}.$$

So we form

$$U_\xi(g) = \sum_{p \in N(k)\backslash P(k)} W_\xi(pg) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right).$$

Proposition 10.1. *$U_\xi(g)$ converges absolutely and uniformly for g in compact subsets, is left invariant under $P(k)$, and its restriction to $GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})$ is rapidly deceasing (modulo the center).*

Note that is $\xi = \varphi$ was indeed a cusp form, this would be its Fourier expansion.

We can make a similar construction for any mirabolic subgroup and to utilize our functional equation we will need to do this. To this end, let Q be the opposite mirabolic

$$Q = \mathrm{Stab}_{GL_n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \left\{ q = \begin{pmatrix} * & \cdots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & 0 \\ * & \cdots & * & 1 \end{pmatrix} \right\}$$

and let $\alpha = \begin{pmatrix} & 1 \\ I_{n-1} & \end{pmatrix}$, a permutation matrix. Then set

$$V_\xi(g) = \sum_{q \in N'(k)\backslash Q(k)} W_\xi(\alpha q g) \quad \text{where} \quad N' = \alpha^{-1} N \alpha.$$

This is again absolutely convergent, uniformly on compact subsets, left invariant under $Q(k)$, and rapidly decreasing (mod center) upon restriction to $GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})$.

Since $P(k)$ and $Q(k)$ together generate $GL_n(k)$, it suffices to show that $U_\xi(g) = V_\xi(g)$, for then

$$\xi \mapsto U_\xi(g) \quad \text{embeds} \quad V_\pi \hookrightarrow \mathcal{A}_0^\infty.$$

We will obtain this equality from the analytic properties of $L(s, \pi \times \pi')$.

Let $V_{\pi'} \subset \mathcal{A}^\infty(GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A}))$ be *any* irreducible subspace of the space of smooth automorphic forms on GL_{n-1}. For example π'

could be cuspidal. We call such π' *proper* automorphic representations. They consist of spaces of automorphic forms.

If $\varphi' \in V_{\pi'}$ then we can form

$$I(s, U_\xi, \varphi') = \int_{GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})} U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det h|^{s-\frac{1}{2}} \, dh$$

and show that this converges for $Re(s) >> 0$. This will then factor in the usual manner into

$$I(s, U_\xi, \varphi') = \prod_v \Psi(s, W_{\xi_v}, W'_{\varphi'_v}).$$

Suppose first that π' is cuspidal. Let T be the finite set of places, containing the archimedean ones, such that π_v, π'_v, and ψ_v are all unramified for $v \notin T$. Then as before

$$\begin{aligned} I(s, U_\xi, \varphi') &= \left(\prod_{v \in T} \Psi(s, W_{\xi_v}, W'_{\varphi'_v}) \right) L^T(s, \pi \times \pi') \\ &= \left(\prod_{v \in T} e(s, W_{\xi_v}, W'_{\varphi'_v}) \right) L(s, \pi \times \pi'). \end{aligned}$$

From our local theory we know that the factors $e_v(s)$ are entire and by assumption $L(s, \pi \times \pi')$ is entire. Hence $I(s, U_\xi, \varphi')$ extends to an entire function of s.

If π' is not cuspidal, then by Langlands' Theorem given last lecture we know that π' is a constituent, and in fact a sub-representation, of an induced representation $\Xi = Ind(\tau_1 \otimes \cdots \otimes \tau_r)$ with each τ_i a cuspidal representation of some GL_{n_i} with $n_i < n - 1$. So each $L(s, \pi \times \tau_i)$ is nice and we can use these to reach the same conclusion, namely that $I(s, U_\xi, \varphi')$ is entire for any $\varphi' \in V_{\pi'}$ for any proper π'.

Similarly if we form

$$I(s, V_\xi, \varphi') = \int_{GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})} V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det h|^{s-\frac{1}{2}} \, dh$$

then this will converge for $Re(s) << 0$, unfolds to

$$I(s, V_\xi, \varphi') = \left(\prod_{v \in T} \widetilde{e}(1 - s, R(w_{n,n-1}\widetilde{W}_{\xi_v}, \widetilde{W}'_{\varphi'_v}) \right) L(1 - s, \widetilde{\pi} \times \widetilde{\pi}'),$$

and continues to an entire function of s.

If we now apply the assumed global functional equation for either $L(s, \pi \times \pi')$ or $L(s, \pi \times \tau_i)$ and the local functional equations for $v \in T$ we may conclude that

$$I(s, U_\xi, \varphi') = I(s, V_\xi, \varphi') \quad \text{for all} \quad \varphi' \in V_{\pi'} \subset \mathcal{A}^\infty(GL_{n-1}).$$

Then an application of the Phragmen–Lindelöf principle implies that these functions are bounded in vertical strips of finite width.

Thus we have

$$\int U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det h|^{s-\frac{1}{2}} \, dh = \int V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det h|^{s-\frac{1}{2}} \, dh$$

with the integration over $GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})$. Using the boundedness in vertical strips, we can apply Jacquet-Langlands' version of Mellin inversion to obtain

$$\int U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) \, dh = \int V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) \, dh$$

now with the integration over $SL_{n-1}(k)\backslash SL_{n-1}(\mathbb{A})$. Then using the weak form of Langlands spectral theory for $SL_{n-1}(k)\backslash SL_{n-1}(\mathbb{A})$ we can conclude that the functions φ' are "complete" and that

$$U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} = V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \quad \text{for} \quad h \in SL_{n-1}(\mathbb{A}), \; \xi \in V_\pi$$

and in particular

$$U_\xi(I_n) = V_\xi(I_n) \quad \text{for all} \quad \xi \in V_\pi.$$

10.3. **Proof of Theorem 10.1 (i).** To conclude the proof of part (i) of Theorem 10.1 we just note that since we have

$$U_\xi(I_n) = V_\xi(I_n) \quad \text{for all} \quad \xi \in V_\pi$$

then for any $g \in GL_n(\mathbb{A})$ we have

$$U_\xi(g) = U_{\pi(g)\xi}(I_n) = V_{\pi(g)\xi}(I_n) = V_\xi(g).$$

So the map $\xi \mapsto U_\xi$ maps $V_\pi \to \mathcal{A}^\infty$. Since U_ξ is given by a Fourier expansion

$$U_\xi(g) = \sum_{N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right)$$

we can compute a non-zero Fourier coefficient to conclude that $U_\xi \not\equiv 0$, and hence the map is injective, and explicitly show that all unipotent periods are zero, and hence that U_ξ is in fact cuspidal. Thus we have $V_\pi \hookrightarrow \mathcal{A}_0^\infty$ as desired.

[To Be Continued]