9. Global L-functions

We return now to the global setting. So once again k is a number field and $\mathbb{A}$ its ring of adeles. Let Σ denote the set of all places of k. Take $\psi : k\backslash\mathbb{A} \to \mathbb{C}^\times$ a non-trivial continuous additive character.

Let (π, V_π) be a unitary smooth cuspidal representation of $GL_n(\mathbb{A})$, which then decomposes as $\pi \simeq \otimes'\pi_v$. Similarly, $(\pi', V_{\pi'})$ will be a unitary smooth cuspidal representation of $GL_m(\mathbb{A})$ with $\pi' \simeq \otimes'\pi'_v$. We will mainly concentrate on the case of $m < n$. The case of $m = n$ can then be worked out as an exercise.

For each place $v \in \Sigma$ we have defined local L- and ε-factors

$$L(s, \pi_v \times \pi'_v) \quad \text{and} \quad \varepsilon(s, \pi_v \times \pi'_v, \psi_v).$$

We then define the global L-function and ε-factor as Euler products.

Definition 9.1. *The global L-function and ε-factors for π and π' are*

$$L(s, \pi \times \pi') = \prod_{v \in \Sigma} L(s, \pi_v \times \pi'_v)$$

and

$$\varepsilon(s, \pi \times \pi') = \prod_{v \in \Sigma} \varepsilon(s, \pi_v \times \pi'_v, \psi_v).$$

Implicit in this definition is the convergence of the products in a half plane $Re(s) >> 0$ and the independence of the ε-factor from the choice of ψ. We will address this below. Then we will turn to showing these L-functions are *nice*. Our scheme will be to relate these Euler products to our global integrals and deduce the global properties of the L-functions from those of our global integrals.

Throughout, we will take $S \subset \Sigma$ to be a finite set of places, containing the archimedean places, such that for all $v \notin S$ we have that π_v, π'_v, and ψ_v are all unramified. The set S can vary, but it should always have these properties.

9.1. **Convergence.**

Choose cusp forms $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ such that under the decomposition $V_\pi \simeq \otimes V_{\pi_v}$ we have $\varphi \simeq \otimes\xi_v$ and similarly $\varphi' \simeq \otimes\xi'_v$. Choose S as above such that for all $v \notin S$, $\xi_v = \xi_v^\circ$ is the K_v-fixed vector in V_{π_v} and similarly $\xi'_v = \xi_v'^\circ$. Then we know from

Lecture 5 that

$$I(s,\varphi,\varphi') = \Psi(s,W_\varphi,W'_{\varphi'}) = \prod_{v\in\Sigma}\Psi(s,W_{\xi_v},W'_{\xi'_v})$$

and this converges absolutely for $Re(s)>1$. By our unramified calculation of Lecture 7 we know that for $v\notin S$ we have

$$\Psi(s,W_{\xi_v^\circ},W'_{\xi_v'^\circ}) = L(s,\pi_v\times\pi'_v).$$

Hence

$$I(s,\varphi,\varphi') = \left(\prod_{v\in S}\Psi(s,W_{\xi_v},W'_{\xi'_v})\right)L^S(s,\pi\times\pi')$$

where $L^S(s,\pi\times\pi')$ is the partial L-function

$$L^S(s,\pi\times\pi') = \prod_{v\notin S}L(s,\pi_v\times\pi'_v).$$

Thus the Euler product for $L^S(s,\pi\times\pi')$ converges for $Re(s)>>0$ and hence

- $L(s,\pi\times\pi')$ converges for $Re(s)>>0$.

Thus our global L-function is well defined.

We could have also deduced the convergence of the infinite product from the Jacquet-Shalika bounds on the Satake parameters for unramified representations. As was pointed out, this would give convergence for $Re(s)>\frac{3}{2}$. In fact, with a bit more work than I have done here, Jacquet and Shalika show absolute convergence (and hence non-vanishing) for $Re(s)>1$.

As for the ε-factor, again from our unramified calculation of Lecture 7 we know that $\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\equiv 1$ for $v\notin S$. So

$$\varepsilon(s,\pi\times\pi') = \prod_{v\in S}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)$$

is only a finite product. From the shape of the local ε-factors from Lectures 6 and 8, we know that it has the form

$$\varepsilon(s,\pi\times\pi') = WN^{\frac{1}{2}-s}$$

with N a positive integer.

The independence of $\varepsilon(s,\pi\times\pi')$ from the choice of ψ can be seen either by investigating how the local ε-factors vary as we vary ψ, which can be done through the local integrals, or as a consequence of the global functional equation below.

9.2. **Meromorphic continuation.** We continuation analyzing the relation between $L(s, \pi \times \pi')$ and our global integrals from above. We have

$$\begin{aligned} I(s, \varphi, \varphi') &= \left(\prod_{v \in S} \Psi(s, W_{\xi_v}, W'_{\xi'_v}) \right) L^S(s, \pi \times \pi') \\ &= \left(\prod_{v \in S} \frac{\Psi(s, W_{\xi_v}, W'_{\xi'_v})}{L(s, \pi_v \times \pi'_v)} \right) L(s, \pi \times \pi') \\ &= \left(\prod_{v \in S} e(s, W_{\xi_v}, W'_{\xi'_v}) \right) L(s, \pi \times \pi') \end{aligned}$$

From our analysis of the global integrals, we know that $I(s, \varphi, \varphi')$ is entire (or $I(s, \varphi, \varphi', \Phi)$ is meromorphic if $m = n$). For each $v \in S$ we have seen that the local ratios $e(s, W_{\xi_v}, W'_{\xi'_v})$ are entire. Since S is a finite set, we can conclude

- $L(s, \pi \times \pi')$ extends to a meromorphic function of s.

9.3. **Poles of L-functions.** In our analysis of the local L-functions in Lectures 6 and 8 we have shown not only that the local ratios $e(s, W_{\xi_v}, W'_{\xi'_v})$ are entire, but in fact that for every $s_0 \in \mathbb{C}$ there is a choice of local Whittaker functions W_v and W'_v such that the ratio $e(s_0, W_v, W'_v) \neq 0$. So as we vary $W_v \in \mathcal{W}(\pi_v, \psi_v)$ and $W'_v \in \mathcal{W}(\pi'_v, \psi_v^{-1})$ we obtain that the poles of the global L-function $L(s, \pi \times \pi')$ are *precisely* those that occur for the families of global integrals

$$\{I(s, \varphi, \varphi')\} \quad \text{or} \quad \{I(s, \varphi, \varphi' \Phi)\}.$$

Hence

- If $m < n$ then $L(s, \pi \times \pi')$ is entire.

- If $m = n$ then $L(s, \pi \times \pi')$ has simple poles precisely at those $s = i\sigma$ and $s = 1 + i\sigma$ with $\sigma \in \mathbb{R}$ such that $\widetilde{\pi} \simeq \pi' \otimes |\det|^{i\sigma}$.

In particular,

- $L(s, \pi \times \widetilde{\pi})$ has simple poles at $s = 0, 1$

- $L(s, \pi \times \widetilde{\pi}')$ has a pole at $s = 1$ iff $\pi \simeq \pi'$.

9.4. **The global functional equation.** We know that our global integrals satisfy a functional equation

$$I(s, \varphi, \varphi') = \widetilde{I}(1 - s, \widetilde{\varphi}, \widetilde{\varphi}').$$

Furthermore, as above, we have decompositions

$$I(s, \varphi, \varphi') = \left(\prod_{v \in S} e(s, W_{\xi_v}, W'_{\xi'_v}) \right) L(s, \pi \times \pi')$$

and

$$\widetilde{I}(1 - s, \widetilde{\varphi}, \widetilde{\varphi}') = \left(\prod_{v \in S} \widetilde{e}(1 - s, R(w_{n,m})\widetilde{W}_{\xi_v}, \widetilde{W}'_{\xi'_v}) \right) L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

By the local functional equations, for each $v \in S$ we have

$$\widetilde{e}(1 - s, R(w_{n,m})\widetilde{W}_{\xi_v}, \widetilde{W}'_{\xi'_v}) = \omega_{\pi'_v}(-1)^{n-1} \varepsilon(s, \pi_v \times \pi'_v, \psi_v) e(s, W_{\xi_v}, W'_{\xi'_v}).$$

We now take the product of both sides over those $v \in S$. Note that since everything is unramified for $v \notin S$, we have

$$\prod_{v \in S} \omega_{\pi'_v}(-1)^{n-1} = \prod_{v \in \Sigma} \omega_{\pi'_v}(-1)^{n-1} = \omega_{\pi'}(-1) = 1$$

and as we have seen above

$$\prod_{v \in S} \varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \prod_{v \in \Sigma} \varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \varepsilon(s, \pi \times \pi').$$

Thus when we take this product we find

$$\prod_{v \in S} \widetilde{e}(1 - s, R(w_{n,m})\widetilde{W}_{\xi_v}, \widetilde{W}'_{\xi'_v}) = \varepsilon(s, \pi \times \pi') \prod_{v \in S} e(s, W_{\xi_v}, W'_{\xi'_v})$$

so that

$$\widetilde{I}(1 - s, \widetilde{\varphi}, \widetilde{\varphi}') = \left(\prod_{v \in S} e(s, W_{\xi_v}, W'_{\xi'_v}) \right) \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

If we combine this with our functional equation for the global integrals, we find our global functional equation

- $L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi}')$.

Note that this equality implies that $\varepsilon(s, \pi \times \pi')$ is independent of ψ.

9.5. **Boundedness in vertical strips.** This is not as simple as it should be. Here is the paradigm. We include the case of $m = n$.

For $v \notin S$ we have

$$L(s, \pi_v \times \pi'_v) = \begin{cases} \Psi(s, W_v^\circ, W_v'^\circ, \Phi_v^\circ) & m = n \\ \Psi(s, W_v^\circ, W_v'^\circ) & m < n \end{cases}.$$

For non-archimedean $v \in S$ there are finite collections $\{W_{v,i}\}$, $\{W'_{v,i}\}$, and $\{\Phi_{v,i}\}$ if necessary such that

$$L(s, \pi_v \times \pi'_v) = \begin{cases} \sum_i \Psi(s, W_{v,i}, W'_{v,i}, \Phi_{v,i}) & m = n \\ \sum_i \Psi(s, W_{v,i}, W'_{v,i}) & m < n \end{cases}.$$

For archimedean places v only if $m = n$ or $m = n-1$ do we know that there are finite families of either smooth or even K_v-finite Whittaker functions $\{W_{v,i}\}$ and $\{W'_{v,i}\}$, and if necessary Schwartz functions $\{\Phi_{v,i}\}$ such that

$$L(s, \pi_v \times \pi'_v) = \begin{cases} \sum_i \Psi(s, W_{v,i}, W'_{v,iy}, \Phi_{v,i}) & m = n \\ \sum_i \Psi(s, W_{v,i}, W'_{v,i}) & m = n-1 \end{cases}.$$

Hence if $m = n$ or $m = n-1$ the there are finite collections of cusp forms $\{\varphi_i\} \subset V_\pi$ and $\{\varphi'_i\} \subset V_{\pi'}$ and if necessary Schwartz functions $\{\Phi_i\} \subset \mathcal{S}(\mathbb{A}^n)$ such that

$$L(s, \pi \times \pi') = \begin{cases} \sum_i I(s, \varphi_i, \varphi'_i, \Phi_i) & m = n \\ \sum_i I(s, \varphi_i, \varphi'_i) & m = n-1. \end{cases}$$

Now boundedness in vertical strips of the L-function $L(s, \pi \times \pi')$ follows from that of the global integrals.

If $m < n-1$ then at the archimedean places we must pass to the topological product $V_{\pi_v} \hat{\otimes} V_{\pi'_v}$ in order to obtain $L(s, \pi_v \times \pi'_v)$, that is,

$$L(s, \pi_v \times \pi'_v) = \Psi(s, W) \quad \text{for} \quad W \in \mathcal{W}(\pi_v \hat{\otimes} \pi'_v, \psi_v).$$

To make our paradigm work we should re-develop the analysis of our global integrals for cusp forms $\varphi(g, h) \in V_\pi \hat{\otimes} V_{\pi'}$, which is a smooth cuspidal representation of the product $GL_n(\mathbb{A}) \times GL_m(\mathbb{A})$. Then we would obtain an equality

$$L(s, \pi \times \pi') = I(s, \varphi) \quad \text{with} \quad \varphi \in V_\pi \hat{\otimes} V_{\pi'}$$

and would then have boundedness in vertical strips as before. There seems to be no obstruction to carrying this out and we hope to soon write up the details. This then gives boundedness in vertical strips in general.

If this makes you nervous, Gelbart and Shahidi have proven boundedness in vertical strips for a wide class of L-functions, including ours, via the Langlands-Shahidi method of analyzing L-functions through the Fourier coefficients of Eisenstein series.

So, no matter how you cut it,

- $L(s, \pi \times \pi')$ is bounded in vertical strips of finite width.

9.6. **Summary.** If we combine these results, we obtain a statement of the basic analytic properties of out L-functions.

Theorem 9.1. *If π is a unitary cuspidal representation of $GL_n(\mathbb{A})$ and π' is a unitary cuspidal representation of $GL_m(\mathbb{A})$ with $m < n$ then $L(s, \pi \times \pi')$ is* **nice**, *i.e.,*

(i) *$L(s, \pi \times \pi')$ converges for $Re(s) >> 0$ and extends to an entire function of s;*
(ii) *this extension is bounded in vertical strips of finite width;*
(iii) *it satisfies the functional equation*

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

In the case of $m = n$ we have a similar result.

Theorem 9.2. *If π and π' are two unitary cuspidal representation of $GL_n(\mathbb{A})$ then*

(i) *$L(s, \pi \times \pi')$ converges for $Re(s) >> 0$ and extends to a meromorphic function of s with simple poles at those $s = i\sigma$ and $s = 1 + i\sigma$ such that $\widetilde{\pi} \simeq \pi' \otimes |\det|^{i\sigma}$; if there are no such $i\sigma$ then $L(s, \pi \times \pi')$ is entire;*
(ii) *this extension is bounded in vertical strips of finite width (away from its poles);*
(iii) *it satisfies the functional equation*

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

9.7. **Strong Multiplicity One revisited.** We will now present the analytic proof of the Strong Multiplicity One Theorem due to Jacquet and Shalika. It is based on the analytic properties of L-functions. First, recall the statement.

Theorem 9.3 (Strong Multiplicity One for GL_n)**.** *Let (π_1, V_{π_1}) and (π_2, V_{π_2}) be two cuspidal representations of $GL_n(\mathbb{A})$. Decompose them as $\pi_1 \simeq \otimes' \pi_{1,v}$ and $\pi_1 \simeq \otimes' \pi_{2,v}$. Suppose that there is a finite set of places S such that $\pi_{1,v} \simeq \pi_{2,v}$ for all $v \notin S$. Then $(\pi_1, V_{\pi_1}) = (\pi_2, V_{\pi_2})$.*

Without loss of generality we may assume π_1 and π_2 are unitary. We know from Section 3 or Theorem 9.2 that $L(s, \pi_1 \times \widetilde{\pi}_2)$ has a pole at $s = 1$ iff $\pi_1 = \pi_2$.

Let us write

$$L(s, \pi_1 \times \widetilde{\pi}_2) = \left(\prod_{v \in S} L(s, \pi_{1,v} \times \widetilde{\pi}_{2,v}) \right) L^S(s, \pi_1 \times \widetilde{\pi}_2).$$

The local L-functions for $v \in S$ are all of the form

$$L(s, \pi_{1,v} \times \widetilde{\pi}_{2,v}) = \begin{cases} P_v(q_v^{-s})^{-1} & v < \infty \\ \prod \Gamma_v(s + *) & v | \infty \end{cases}.$$

So in either case they are never zero. We also know from Lectures 6 and 8 that the local integrals are absolutely convergent for $Re(s) \geq 1$. So the local L-factors can have no poles in this region either. Hence the finite product

$$\prod_{v \in S} L(s, \pi_{1,v} \times \widetilde{\pi}_{2,v})$$

has no zeros or poles in $Re(s) \geq 1$. Thus $L(s, \pi_1 \times \widetilde{\pi}_2)$ has a pole at $s = 1$ iff $L^S(s, \pi_1 \times \widetilde{\pi}_2)$ does.

Since $\pi_{1,v} \simeq \pi_{2,v}$ for all $v \notin S$ we have

$$L^S(s, \pi_1 \times \widetilde{\pi}_2) = L^S(s, \pi_1 \times \widetilde{\pi}_1).$$

By the same argument as above, $L^S(s, \pi_1 \times \widetilde{\pi}_1)$ will have a pole at $s = 1$ since the full L-function $L(s, \pi_1 \times \widetilde{\pi}_1)$ does, again by Theorem 9.2.

Thus $L(s, \pi_1 \times \widetilde{\pi}_2)$ does indeed have a pole at $s = 1$ and so $\pi_1 \simeq \pi_2$. Then Multiplicity One for GL_n gives that in fact $(\pi_1, V_{\pi_1}) = (\pi_2, V_{\pi_2})$.

9.8. **Generalized Strong Multiplicity One.** Jacquet and Shalika were able to push this technique further to obtain a version of the Strong Multiplicity One Theorem for non-cuspidal representations. To state it, we must first recall a theorem of Langlands.

If π is any irreducible automorphic representation of $GL_n(\mathbb{A})$ then there exists a partition $n = n_1 + \cdots + n_r$ of n and cuspidal representations τ_i of $GL_{n_i}(\mathbb{A})$ such that π is a constituent of the induced representation

$$\Xi = \mathrm{Ind}_{Q(\mathbb{A})}^{GL_n(\mathbb{A})}(\tau_1 \otimes \cdots \otimes \tau_r).$$

Langlands worked in the context of K-finite automorphic representations, but the result is valid for smooth automorphic representations as well. It is a consequence of the theory of Eisenstein series. Similarly, if π' is another automorphic representation of $GL_n(\mathbb{A})$ then π' will be a constituent of a similarly induced representation

$$\Xi' = \mathrm{Ind}_{Q'(\mathbb{A})}^{GL_n(\mathbb{A})}(\tau_1' \otimes \cdots \otimes \tau_{r'}')$$

associated to a second partition $n = n_1' + \cdots + n_{r'}'$.

Theorem 9.4 (Generalized Strong Multiplicity One)**.** *Let π and π' be two automorphic representations of $GL_n(\mathbb{A})$ as above. Suppose that there is a finite set of places S such that $\pi_v \simeq \pi_v'$ for all $v \notin S$. Then $r = r'$ and there is a permutation σ of $\{1, \ldots, r\}$ such that $n_i = n_{\sigma(i)}'$ and $\tau_i \simeq \tau_{\sigma(i)}$.*

Thus the knowledge of the local components of π at almost all places completely determines the "cuspidal support" of π. In particular, the "cuspidal support" of π is well defined. As a consequence of this result Jacquet and Shalika showed the existence of the category of *isobaric representations* for $GL_n(\mathbb{A})$.

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