Coding into Ramsey sets

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November 15, 2002

*This Research was supported through a European Community Marie Curie Fellowship

Classical Ramsey Theory

Theorem 1. *Every Analytic set of* $[\mathbb{N}]^\infty$ *is Ramsey, i.e., either it contains a cube* $[A]^\infty$ *or its complement contains a cube.*

The original proof of Galvin-Prikry of that all Borel sets of $[\mathbb{N}]^\infty$ are Ramsey cannot be extended to the analytic case. The proof of this case from Silver or Ellentuck involve very different ideas (forcing, absoluteness and topological sense of the Ramsey property).

Several uses (for example Farahat proof of Rosenthal's Theorem using that closed sets are Ramsey). There are many others cases when the full strength is needed.

Gowers' Block Ramsey Theorem for Banach spaces

From now on we fix an infinite dimensional Banach space $E = (E, \|\cdot\|)$ with a fixed Schauder basis $(e_n)_n$. We can assume that the basis is bimonotone.

$\boldsymbol{B_1} = \boldsymbol{B_1}(E)$ is the set of infinite sequences $(x_n)_n$ of normalized block vectors, i.e., for any n, x_n has finite support $\operatorname{supp} x_n = \{m : \langle x_n, e_m\rangle \neq 0\}$ and $\max \operatorname{supp} x_n < \min \operatorname{supp} x_{n+1}$. We will call them block sequences. $\boldsymbol{B_1}$ is closed as a subset of the cartesian product $E^{\mathbb{N}}$ and hence it is a Polish space.

Recall that subsets of Polish spaces can be classified according to their topological complexity (Borel, Analytic, Co-analytic...)

For a finite block sequence $s = (s_0, ..., s_k)$ (which can be empty) and an infinite one $A = (a_n)_n$, let $[s; A]$ be the set of block sequences $s^\frown(b_n)_n = (s_0, ..., s_k, b_0, b_1, ...)$ such that for all n, b_n is in the unit sphere $S(A)$ of A. We denote $[\emptyset; A]$ by $[A]$.

Let the D-topology in $\boldsymbol{B_1}$ be that with basic open sets $[(s_0, ..., s_k); (e_n)_n]$.

Given a set of block sequences σ and $s < A$, we say that σ **is large for** $[s; A]$ iff for every $B \in [A]$ there is some $C \in [B]$ such that $s^\frown C \in \sigma$.

We say that σ is **strategically large for** $[s; A]$ iff: i.e. Player II has a winning strategy for $\sigma[s; A]$, for E not containing c_0.

The game looks like this:

$$\begin{array}{c||cccccc}
\mathbf{I} & Y_1 \in [Y] & & \cdots & Y_n \in [Y] & & \cdots \\ \hline
\mathbf{II} & & y_1 \in Y_1 & \cdots & & y_n \in Y_n & \cdots
\end{array}$$

There are two players, I and II. I starts playing $Y_1 \in [Y]$, then II chooses $y_1 \in Y_1$. Then player I plays $Y_2 \in [Y]$, and II chooses $y_2 \in Y_2$ with $y_2 > y_1$, and so on. II wins the game iff $(y_n)_n \in \sigma$. Otherwise I wins.

For a decreasing sequence of positive reals $\Delta = (\delta_n)_n$, and $\sigma \subseteq \boldsymbol{B_1}$, let σ_Δ be the set of block sequences $Y = (y_n)_n$ such that there is some $X = (x_n)_n \in \sigma$ such that for all n, $\|x_n - y_n\| \leq \delta_n$ (in short, by $d(X, Y) \leq \Delta$). σ^o_Δ will be the same, except that now we change $\leq$ by $<$.

Definition 1. A set σ is **Ramsey** iff for every $s < A$, every Δ the following dichotomy holds: There is some $B \in [A]$ such that either $\sigma \cap [s; A] = \emptyset$ or $[s, B] \subseteq \sigma_\Delta$.

Using distortion, it is quite easy to show that if E does not contain c_0, there are open sets that are not Ramsey.

For c_0 the situation is different.

Theorem 2. [Gowers] *Every analytic set of* $\boldsymbol{B_1}(c_0)$ *is Ramsey.*

Definition 2. A set σ is **weakly Ramsey** iff for every $s < A$, every Δ the following dichotomy holds: Either $\sigma \cap [s; A] = \emptyset$ or there is some $B \in [A]$ such that σ_Δ is strategically large for $[s; B]$.

Let $\boldsymbol{\mathcal{G}}$ be the family of weakly- Ramsey sets. W.l.o.g. we will always assume that all $\Delta = (\delta_n)_n$ are decreasing sequences of positive reals less than 1.

Theorem 3. [Gowers] *Let E be a Banach space, and let σ be an analytic set of (normalized) block sequences which is large for E, and let $\Delta > 0$. Then there is a block sequence Y such that σ_Δ is strategically large for Y.*

Question Is there any application which needs the full strengh of the Gowers' Theorem?

Coding information into weakly-Ramsey sets

Fact

The coloring $x \mapsto$ *sign of the last non zero coordinate of* x, defined for x finitely supported vector, is **not** monochromatic in any finite dimensional block subspace.

The same situation occurs for example, if we have in the space two separated asymptotic sets A_0 and A_1: given a large enough finite block sequence $F = (x_i)_{i=1}^n$ we can colour all block vectors of the unit sphere of the span on F according if they are in A_0 or not.

We will show that the absence of the true pigeonhole principle in the block Ramsey theory for Banach spaces leads to some dramatic differences between that theory and the classical Ramset theory.

Denote by $\mathcal{N}^{\uparrow}$ the set of strictly increasing sequences of positive integers with the topology as a subset of the Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$. It is well know that any analytic (in particular Borelian) set σ of $\boldsymbol{B_1}$ is the projection of a closed set of $\boldsymbol{B_1} \times \mathcal{N}^{\uparrow}$.

Fix any $A \subseteq \boldsymbol{B_1} \times \mathcal{N}^{\uparrow}$, let $\sigma = P_0(A)$ be the image under the first projection $P_0 : (X, \vec{\alpha}) \mapsto X$. So, for a block sequence X, $X \in \sigma$ iff there is some $(\alpha_n)_n \in \mathcal{N}^{\uparrow}$ such that $(X, (\alpha_n)_n) \in \boldsymbol{B_1} \times \mathcal{N}^{\uparrow}$.

We are going to code any pair $((x_n)_n, (\alpha_n)_n))$ with a single block sequence $(x_n)_n * (y_n)_n = (x_0, y_0, x_1, y_1, ...)$ using an anti-Ramsey property:

$$\begin{array}{cccccccc} y_0 & y_1 & \cdots & y_{\alpha_0-1} & y_{\alpha_0} & \cdots & y_{\alpha_1-1} & \cdots \\ - & - & \cdots & + & - & \cdots & + & \cdots \end{array}$$

Where the signs $\mathsf{sgn}(y_i)$ are the sign of

$$e^*_{\max \mathsf{supp}\ y_i} y_i.$$

Or if we have two asymptotic sets A_0, A_1 separated $(d(A_0, A_1) > 0)$

$$\begin{array}{cccccccc} y_0 & y_1 & \cdots & y_{\alpha_0 - 1} & y_{\alpha_0} & \cdots & y_{\alpha_1 - 1} & \cdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 & \cdots \end{array}$$

Where $\varepsilon_i = 0$ iff $y_i \in A_0$, and now for all i, $y_i \in A_0 \cup A_1$, and there is infinitely many i such that $y_i \in A_1$.

Let $\mathcal{B}_{\pm}$ be the set of block sequences $(x_n)_n$ such that there are infinitely many n such that sign$x_{2n+1} = +$. This is the set of block sequences that code some pair $(Y, \vec{\alpha})$.

Let $\tilde{\Lambda}_{\pm} : \boldsymbol{B_1} \to \boldsymbol{B_1} \times \{0,1\}^{\mathbb{N}}$ be the decoder mapping, $\Lambda_{\pm} X = (X_e, (\varepsilon_n)_n) \in \boldsymbol{B_1} \times \{0,1\}^{\mathbb{N}}$ where for $X = (x_n)_n$, $X_e = (x_{2n})_n$ and $\{\varepsilon_n\}_n = 0$ iff sgn$x_{2n+1} = -$.

Proposition 1. *$\tilde{\Lambda}_{\pm}$ is of Baire class 1, i.e., the preimage under Λ of an open set of $\boldsymbol{B_1} \times \{0,1\}^{\mathbb{N}}$ is a countable union of closed sets.*

As a consequence, $\mathcal{B}_\pm = \tilde{\Lambda}_\pm^{-1} N$, where N is the G_δ natural copy of $\mathcal{N}^\uparrow$ in $\{0,1\}^{\mathbb{N}}$. Therefore, $\mathcal{B}_\pm$ is $F_{\sigma\delta}$ subset of $\boldsymbol{B_1}$.

Let $\Lambda_\pm = \tilde{\Lambda}_\pm | \mathcal{B}_\pm : \mathcal{B}_\pm \to \boldsymbol{B_1} \times N \to \boldsymbol{B_1} \times \mathcal{N}^\uparrow$, i.e. $\Lambda_\pm : \mathcal{B}_\pm \to \boldsymbol{B_1} \times \mathcal{N}^\uparrow$ be the decoder mapping, $\Lambda_\pm X = (X_e, (\alpha_n)_n) \in \boldsymbol{B_1} \times \mathcal{N}^\uparrow$ where for $X = (x_n)_n$, $X_e = (x_{2n})_n$ is the even part of X and $\{\alpha_n\}_n = \{n+1 : \text{sgn} x_{2n+1} = +\}$.

Suppose now that E has two asymptotic separated sets A_0 and A_1. We assume that they are closed and moreover that $A_i \cap \{x \in S_E \ : \ x \text{ is finitely supported}\}$ are also asymptotic, for $i = 0, 1$.

Let $\mathcal{B}_a$ be the set of block sequences such that $X_{\mathsf{odd}} \in \Pi_i^\infty(A_0 \cup A_1)$, and there is infinitely many n such that $x_{2n+1} \in A_1$, where for $X = (x_n)_n$, $X_{\mathsf{odd}} = (x_{2n+1})_n$ is the odd part of X.

Let $\Lambda_a : \mathcal{B}_a \to \boldsymbol{B_1} \times \mathcal{N}^\uparrow$, the decoder mapping $\Lambda_a(x_n)_n = ((x_{2n})_n, (\alpha_n)_n)$, where $\{\alpha_n\} = \{n+1 : x_{2n+1} \in A_1\}$.

Proposition 2.

1. $\mathcal{B}_a$ *is a* G_δ *set. (Proof:* $(x_n)_n \in \mathcal{B}_a$ *iff for all* n $x_{2n+1} \in A_0 \cup A_1$ *and for all* n *there is some* $m \geq n$ *such that* $x_{2m+1} \notin A_0$. *Intersection of a closed set an a* G_δ *set).*

2. $\Lambda_a : \mathcal{B}_a \to \boldsymbol{B_1} \times \mathcal{N}^{\uparrow}$ *is a continuous mapping.*

Given a set $A \subseteq \boldsymbol{B_1} \times \mathcal{N}^{\uparrow}$, let denote $\sigma(A) = P_0 A$, where $P_0 : \boldsymbol{B_1} \times \mathcal{N}^{\uparrow} \to \boldsymbol{B_1}$ is the first projection. Let $\tau_{\pm}(A) = \Lambda_{\pm}^{-1} A$, and $\tau_a(A) = \Lambda_a^{-1} A$.

Notice that $\tau_{\pm}(A)_{\text{even}}, \tau_a(A)_{\text{even}} \subseteq \sigma(A)$.

Proposition 3. *For any block sequence X there is some $Y \in [X]$ such that*

$$\sigma(C) \cap [Y] \subseteq (\tau_{\pm}(C) \cap [X])_{\mathit{even}} \subseteq \sigma(C) \cap [X].$$

Proof Fix X, and check that $Y = X_{\text{even}}$ works.

Proposition 4. *If A_0, A_1 are as before, for any block sequence X there is some $Y \in [X]$ such that*

$$\sigma(C) \cap [Y] \subseteq (\tau_a(C) \cap [X])_{\textit{even}} \subseteq \sigma(C) \cap [X].$$

Proof Fix $(x_n)_n$, and let (n_k) be a subsequence such that for all k, all $i = 0, 1$,

$$A_i \cap S(\langle x_j \rangle_{j=n_k+1}^{n_{k+1}-1}) \neq \emptyset$$

The sequence $Y = (x_{n_k})_k$ works.

Some facts about the codings

Let ε be either $\pm$ or a.

1. c_0 does not admit the coding Λ_a.

2. If E does not contain c_0, there is a block subspace of it with two asymptotic separated subsets, and therefore the coding Λ_a is allowed.

3. $\sigma(A)$ is large iff $\tau_\varepsilon(A)$ is large.

4. For any Δ (strictly decreasing or constant 0), if $(\tau_\varepsilon(A))_\Delta$ is strategically large for $[X]$, then $(\sigma(A))_\Delta$ is strategically large for $[X]$.

5. If $\tau_\varepsilon(C)$ is weakly-Ramsey, then $\sigma(C)$ is weakly-Ramsey.

6. For the case of $E = c_0$, if $\tau_\pm(C)$ is Ramsey, then $\sigma(C)$ is Ramsey.

Theorem 4. *Let $\mathcal{A}$ be a family of subsets of $\boldsymbol{B_1} \times \mathcal{N}^{\uparrow}$. Then, $\tau_a(\mathcal{A}) \subseteq \mathcal{G}_E$ implies that $\sigma(\mathcal{A}) \subseteq \mathcal{G}_E$.*

Corollary 1. *Let E not containing c_0, and let $\mathcal{A}$ the family of closed sets of $\boldsymbol{B_1} \times \mathcal{N}^{\uparrow}$. Then, $G_\delta \subseteq \mathcal{G}_E$ directly implies that all analytic sets are weakly-Ramsey*

Corollary 2. *For the case $E = c_0$, we get that all analytic sets are Ramsey as a immediate consequence that all $F_{\sigma\delta}$ are.*

Corollary 3. *It is not possible to show (in ZFC) that $\mathcal{G}$ or the family of Ramsey subsets of $\boldsymbol{B_1}(c_0)$ are closed under complements. Precisely, If there is a $\underset{\sim}{\Sigma}^1_2$ well-ordering of reals, then $\mathcal{G}$ is not closed under complements.*

Proof If there is a $\underset{\sim}{\Sigma}^1_2$ well ordering of reals, then there is a $\underset{\sim}{\Sigma}^1_2$ non-weakly Ramsey set σ of block sequences in any Banach space. Take a coanalytic set $C \subseteq \boldsymbol{B_1} \times \mathcal{N}^{\uparrow}$ such that $\sigma = \sigma(C)$. Then $\tau_\varepsilon(C)$ is an coanalytic set and not weakly-Ramsey (and hence not Ramsey for the case of c_0).

Notice that the 3 last properties present a behaviour which is very different to that of Ramsey spaces.

G_δ sets

Let $\sigma = \bigcap_n \tau_n$, each τ_n open of $\boldsymbol{B_1}$, and $\Delta > 0$.

The classical procedure to prove that σ is w-Ramsey should be: First one show that all open sets are w-Ramsey and then, using diagonalization, that there is a block sequence X for which $\sigma \cap [X]$ is clopen. However this do not seem possible in the case of $\boldsymbol{B_1}$. The main reason is that could be that

$$(\bigcap_n \tau_n)_\Delta \subsetneq \bigcap_n (\tau_n)_\Delta.$$

In order to prove that σ is w-Ramsey from the fact that all τ_n are w-Ramsey, we have to control why a given block sequence $(x_n)_n$ is in τ_n.

Precisely, We define $\Phi : \sigma \to \mathcal{N}$ by letting $\Phi(X)(n) = k_n$ be such that $B(X|k_n, 1/k_n) \subseteq \tau_n$. For any $\theta \in [\mathbb{N}]^{<\infty}$, let $\sigma_\theta = \Phi^{-1}\langle\theta\rangle$.

We use the family $\{\sigma_\theta\}_{\theta\in[\mathbb{N}]^{<\infty}}$ (a **Suslin scheme**) as in the original proof of Gowers that all analytic sets are w-Ramsey (in that case, given an analytic set σ, we use a continuous onto mapping $f : \mathcal{N} \to \sigma$ to produce the Souslin scheme).

Large Cardinals and w-Ramsey sets

Proposition 5. *Under a suitable large-cardinal assumption every set of infinite block sequences that belongs to* $L(\mathbb{R})$ *is Ramsey (i.e. all definable sets).*

proof Given a definable set σ we use that under this conditions, $\sigma = \bigcup_{\alpha<\aleph_1} \sigma_\alpha$ a 'canonical decomposition', where each σ_α is an analytic set of block sequences (note that each individual set σ_α belongs to $L(\mathbb{R})$, but not necessarily the sequence $(\sigma_\alpha)_{\alpha<\aleph_1}$).

Using a poset $\mathbb{P}$ introduced in (Bagaria, Lopez-Abad '2001) and using Gowers' Theorem for analytic sets, we are able to reduce σ to an open set.