ℓ^1-spreading models in mixed Tsirelson spaces

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Mixed Tsirelson space $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$

$(\theta_n) \subseteq (0,1)$ nonincreasing null sequence, $\theta_{m+n} \geq \theta_m \theta_n$ for all m, n.

$\mathcal{S}_n$ Schreier families

$\mathcal{S}_1 = \{E \subseteq \mathbb{N} : |E| \leq \min E\}$

$\mathcal{S}_{n+1} = \{\cup_{i=1}^{n} E_i : E_i \in \mathcal{S}_n, E_1 < \cdots < E_n, n \leq \min E_1\}$

$\mathcal{S}_\omega = \{E : E \in \mathcal{S}_n \text{ for some } n \leq \min E\}$

$(E_i)_{i=1}^{n}$ is $\mathcal{S}_n$-admissible if $E_1 < \cdots < E_n$ and $\{\min E_i\}_{i=1}^{n} \in \mathcal{S}_n$.

Let (e_n) be the unit vector basis of c_{00}. If $E \subseteq \mathbb{N}$ and $x = \sum a_n e_n$, then $Ex = \sum_{n \in E} a_n e_n$.

$T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$ is the completion of c_{00} with respect to the implicitly defined norm

$$\|x\| = \|x\|_{c_0} \vee \sup_n \theta_n \sup \sum_{i=1}^{k} \|E_i x\|,$$

where the last sup is taken over all $\mathcal{S}_n$-admissible families $(E_i)_{i=1}^{k}$.

ℓ^1-$\mathcal{S}_\alpha$-spreading models

A bounded sequence in a Banach space (x_n) is an ℓ^1-$\mathcal{S}_\alpha$-spreading model if there exists $\delta > 0$ such that
$$\|\sum_{n\in E} a_n x_n\| \geq \delta \sum_{n\in E} |a_n|$$
for all $E \in \mathcal{S}_\alpha$.

Theorem 1. (Argyros, Deliyanni and Manoussakis) *If* $\lim \theta_n^{1/n} = 1$*, then every block subspace of* $T[(\theta_n, \mathcal{S}_n)_{n=1}^\infty]$ *contains an* ℓ^1*-*$\mathcal{S}_\omega$*-spreading model.*

“Non-hereditary” situation

Theorem 2. *If* $\lim_n \limsup_m \theta_{m+n}/\theta_m > 0$*, then every subspace of* $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$ *generated by a subsequence of the unit vector basis contains an* ℓ^1*-*$\mathcal{S}_\omega$*-spreading model.*

Remark. This is the best possible for the “non-hereditary” question, in view of the following observations.

1. (L. & Tang) If $\lim_n \limsup_m \theta_{m+n}/\theta_m = 0$, respectively > 0, then the Bourgain ℓ^1-index of $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$ is ω^ω, respectively $\omega^{\omega\cdot 2}$.
2. (Judd & Odell) The Bourgain ℓ^1-index of any separable Banach space X not containing ℓ^1 is of the form ω^α. The order of any ℓ^1-tree in X is strictly less than the ℓ^1-index.
3. If a block subspace of $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$ contains an ℓ^1-$\mathcal{S}_\alpha$-spreading model, then it contains an ℓ^1-$\mathcal{S}_{\alpha+n}$-spreading model (with different constants) for all n.

The "hereditary" question

Let X be a block subspace of the mixed Tsirelson space $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$. If X contains an ℓ^1-$\mathcal{S}_\omega$-spreading model, then

$$\lim_n \limsup_m \theta_{m+n}/\theta_m > 0 \qquad (\dagger)$$

must hold. On the other hand, if $(\dagger)$ holds and X contains a block equivalent to a subsequence (e_{k_n}) of (e_n), then X contains an ℓ^1-$\mathcal{S}_\omega$-spreading model.

Theorem 3. *The following statements are equivalent.*

1. $(\dagger)$ *holds and X contains a normalized block (x_n) that is equivalent to the sequence (e_{k_n}), where $k_n = \max \operatorname{supp} x_n$,*
2. *X contains an ℓ^1-$\mathcal{S}_\omega$-spreading model,*
3. *X contains ℓ^1-$\mathcal{S}_n$-spreading models with uniform constants,*
4. *The Bourgain ℓ^1-index of X is $\omega^{\omega\cdot 2}$.*

Remark. The theorem of Argyros, Deliyanni and Manoussakis follows because of

Theorem 4. (Judd and Odell) *If X contains an ℓ^1-$\mathcal{S}_{2n}$-spreading model with constant δ, then it contains an ℓ^1-$\mathcal{S}_n$-spreading model with constant $\sqrt{\delta}$.*

The key step is captured in the following proposition. For every n, let $\|\cdot\|_n$ be the equivalent norm on $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$ defined by

$$\|x\|_n = \sup \sum_{i=1}^{k} \|E_i x\|,$$

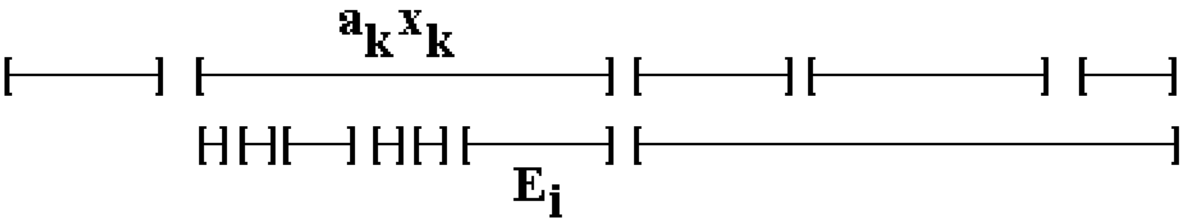
where the sup is taken over all $\mathcal{S}_n$-admissible families $(E_i)_{i=1}^{k}$. If $X = [(z_k)_{k=1}^{\infty}]$ for some block sequence (z_k), the n-tail of X is $[(z_k)_{k=n}^{\infty}]$.

Proposition 5. *Assume that X has the following property:*

($*$) *There exists $C < \infty$ such that for all n, there exists x in the n-tail of X with $\|x\| = 1$ and $\|x\|_n \leq C$.*

Then X contains a normalized block (x_n) that is equivalent to the sequence (e_{k_n}), where $k_n = \max \operatorname{supp} x_n$,

Observe that 4. of Theorem 3 implies ($*$). For all n, 4. of Theorem 3 gives a finite block sequence $(x_k)_{k\in I}$ in the n-tail of X (uniformly) equivalent to the $\ell^1(I)$ basis so that $\{\max \operatorname{supp} x_k : k \in I\}$ is a maximal $\mathcal{S}_{n+1}$ set. Choose scalars (a_k) so that $\sum_I |a_k| = 1$ and $\sum_{k\in F} |a_k|$ is small whenever $\{\max \operatorname{supp} x_k : k \in F\} \in \mathcal{S}_n$. Take $x = \sum_{k\in I} a_k x_k$.



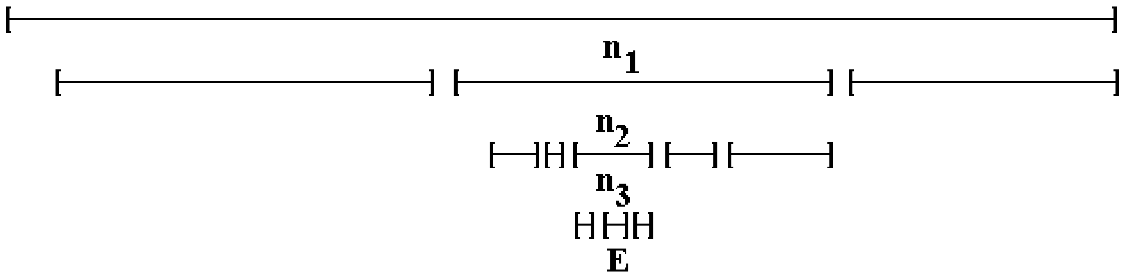
Sum over long E_i's ≤ 1

Sum over short E_i's $\leq \theta_n^{-1} \sum_{k \in F} |a_k|$ for some set F such that $\{\max \operatorname{supp} x_k : k \in F\} \in \mathcal{S}_n$.

Remark. If X has $(*)$, then for all n, there exist $y, z \in X$ such that $\|y\| = \|z\| = 1$, and $\|y\|_n \leq C$, $\|z\|_n \sim 1/\theta_n$. Consequently, if $(*)$ holds hereditarily, then the sequence of norms $(\|\cdot\|_n)$ arbitrarily distorts $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$.

Admissible trees

The norm of an element in $T[(\theta_n, \mathcal{S}_n)_{n=1}^\infty]$ can be computed by means of admissible trees.



- If a node splits, its children form an $\mathcal{S}_n$-admissible collection for some n.
- For the node E shown, we assign it a tag $t(E) = \theta_{n_1}\theta_{n_2}\theta_{n_3}$. We also let $\ell(E) = n_1 + n_2 + n_3$.
- $\mathcal{T}x = \sum t(E)\|Ex\|$, summing over all terminal nodes.
- $\|x\| = \sup \mathcal{T}x$, sup taken over all admissible trees $\mathcal{T}$.
- With respect to a particular element x, we can always choose to split the nodes of an admissible tree until $\|Ex\| = \|Ex\|_{c_0}$ for all terminal nodes E.

Proof of the Proposition. Choose $1 = n_0 < n_1 < \dots$ and a normalized block (x_k) in X so that

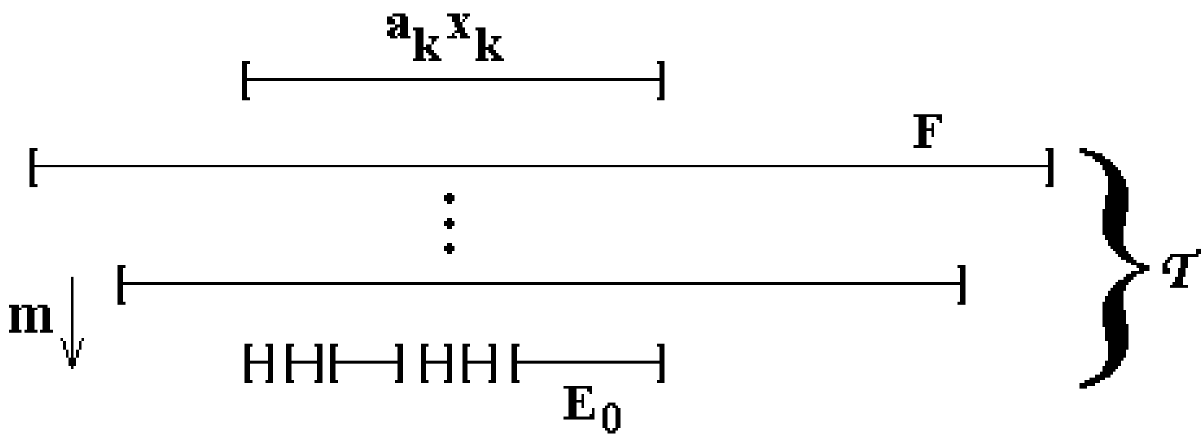
- $\|x_k\|_{n_{k-1}} \le C$,
- $\theta_{n_k} \|x_k\|_{\ell^1} \le 1/2^k$.

We want to show that

$$\|x\| = \|\sum a_k x_k\| \preceq \|\sum a_k e_{j_k}\| = \|y\|,$$

where $j_k = \max \operatorname{supp} x_k$.

Consider an admissible tree $\mathcal{T}$. A node in $\mathcal{T}$ is *short* if it is contained in $\operatorname{supp} x_k$ for some k. We assume that a node is terminal in $\mathcal{T}$ if and only if it is short.



Case I. $m \le n_{k-1}$

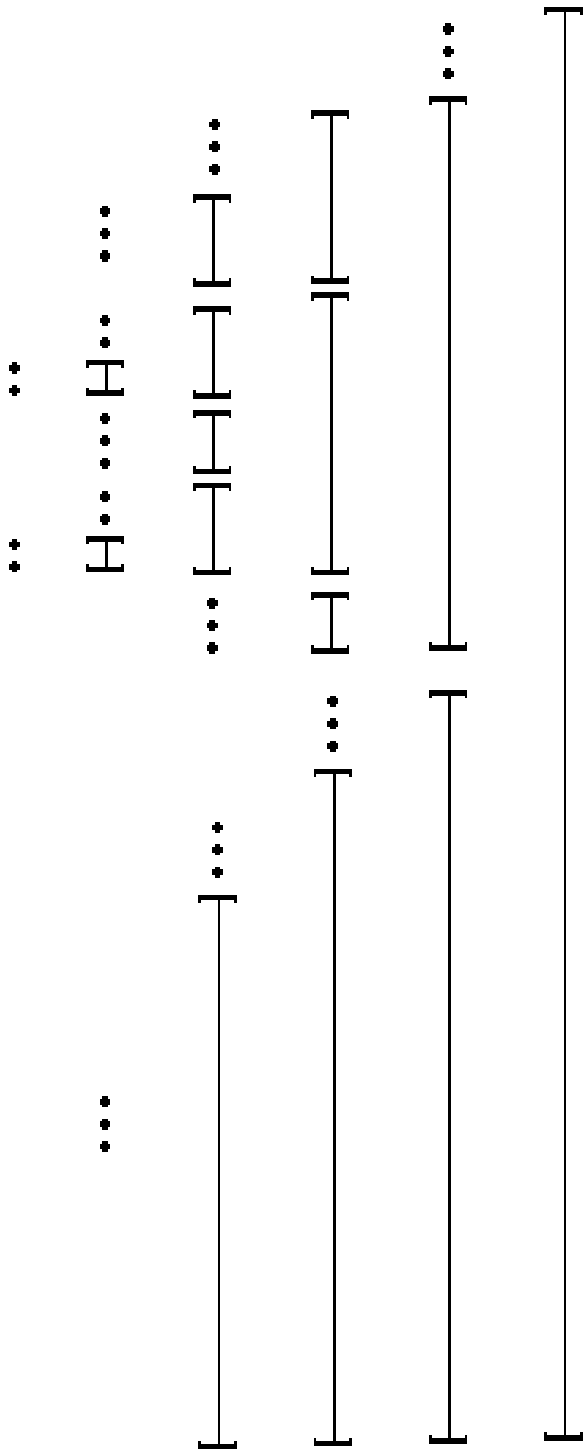
$$\sum_E |a_k| t(E) \|E x_k\| \le C |a_k| t(E_0) = C |a_k| t(E_0) \|E_0 e_{j_k}\|_{c_0}$$

Case II. $\ell(E) \ge n_k$

$$\sum_E |a_k| t(E) \|E x_k\| \le |a_k| \theta_{n_k} \|x_k\|_{\ell^1} \le |a_k|/2^k$$

Case III. $n_{k-1} < m \le \ell(E) < n_k$

Observe that $\sum_E |a_k| t(E) \|Ex_k\| \le |a_k| t(F)$



Let $\mathcal{A}_n$ be the family of all subsets of $\mathbb{N}$ with at most n elements. Define the mixed Tsirelson space $Z = T[(\theta_n, \mathcal{A}_n)_{n=2}^{\infty}]$ with $\mathcal{A}_n$ in place of $\mathcal{S}_n$. The foregoing arguments show:

Theorem 6. *If* $\lim \theta_{2n}/\theta_n = 1$, *then every block subspace of* Z *contains a block sequence equivalent to the unit vector basis* (e_k) *of* Z. *It follows that* Z *is complementably minimal. Moreover, the sequence of norms*

$$\|x\|_n = \sup\{\sum_{r=1}^{n} \|E_r x\| : E_1 < \cdots < E_n\}$$

arbitrarily distorts Z.

These results are due to Schlumprecht.