

# Some Ordinal Indices in Banach Space Theory

The Szlenk index:

If  $X$  is a Banach space,  $A \subset X$ ,  $B \subset X^*$ , and  $\varepsilon > 0$  let

$$P_0(\varepsilon, A, B) = B,$$

for all  $\alpha < \omega_1$ ,

$$P_{\alpha+1}(\varepsilon, A, B) = \left\{ x^* \in X^* : \exists (x_n^*) \in P_\alpha(\varepsilon, A, B), \right. \\ \left. \exists (x_n) \in A, x_n^* \xrightarrow{w^*} x^*, x_n \xrightarrow{w} 0, \text{ and } \lim x_n^*(x_n) \geq \varepsilon \right\}$$

and for limit ordinals,  $\beta$ ,

$$P_\beta(\varepsilon, A, B) = \bigcap_{\alpha < \beta} P_\alpha(\varepsilon, A, B)$$

Usually  $A = B_X$  and  $B = B_{X^*}$ . ( $B_Y = \{y \in Y : \|y\| \leq 1\}$ )

If  $X$  and  $X^*$  are separable, there is an ordinal  $\alpha < \omega_1$ , such that

$$P_\alpha(\varepsilon, B_X, B_{X^*}) \neq \emptyset$$

and

$$P_{\alpha+1}(\varepsilon, B_X, B_{X^*}) = \emptyset.$$

$$\eta(\varepsilon, X) = \alpha + 1$$

is the  $\varepsilon$  Szlenk index of  $X$ .

$$\eta(X) = \sup_{\varepsilon > 0} \eta(\varepsilon, X)$$

is an isomorphic invariant called the Szlenk index of  $X$ . (If the sets are never empty, the Szlenk index is  $\omega_1$ .)

It is not hard to see that

$$x^* \in P_\alpha(\varepsilon, B_X, B_{X^*}) \Rightarrow x^* \in P_{\alpha \cdot k}(\varepsilon/k, B_X, B_{X^*}).$$

Thus the index is always of the form  $\omega^\gamma$ .

The Szlenk index of a space is larger than or equal to the index of both its subspaces and quotient spaces.

**Theorem 1 (Szlenk).** There is no separable, reflexive Banach space  $X$  such that every separable reflexive Banach space is isomorphic to a subspace of  $X$ .

Proof: For every  $\alpha < \omega_1$  there are separable reflexive spaces with Szlenk index larger than  $\alpha$ .

Variants of the Szlenk index:

$$P_{\alpha+1}(\varepsilon, A, B) = \left\{ x^* \in X^* : \exists (x_n^*) \in P_\alpha(\varepsilon, A, B), \right. \\ \left. x_n^* \xrightarrow{w^*} x^*, \lim_n \|x_n^* - x^*\| \geq \varepsilon \right\}$$

If  $X^*$  is separable or  $X$  does not contain a subspace isomorphic to  $\ell_1$ , this change gives a slightly different  $\varepsilon$ -Szlenk index, but  $\eta(X)$  is unchanged.

Convex Szlenk index [Godefroy-Kalton-Lancien]:

$$P_{\alpha+1}(\varepsilon, A, B) = \overline{\text{co}} \left\{ x^* \in X^* : \exists (x_n^*) \in P_\alpha(\varepsilon, A, B), \right. \\ \left. x_n^* \xrightarrow{w^*} x^*, \lim_n \|x_n^* - x^*\| \geq \varepsilon \right\}$$

This was applied in the theory of uniform homeomorphism of Banach spaces and renormings in the finite  $\varepsilon$  index case.

The Szlenk index has been computed for some spaces. A particularly interesting case is that of  $C(K)$  where  $K$  is a countable, compact metric space. Here the index parallels the Mazurkiewicz-Sierpinski classification of the countable compact metric spaces.

Recall that  $K^{(0)} = K$ ,  $K^{(\alpha+1)}$  is the set of limits of non-trivial sequences in  $K^{(\alpha)}$  and  $K^{(\alpha)} = \bigcap_{\beta < \alpha} K^{(\beta)}$  for limit ordinals.

If  $K^{(\gamma)}$  has finite positive cardinality  $n$ , then  $K$  is homeomorphic to  $[1, \omega^\gamma \cdot n]$ .

(I will abbreviate  $C([1, \omega^\gamma \cdot n])$  to  $C(\omega^\gamma \cdot n)$ .)

It is easy to see that for  $0 < \varepsilon \leq 1$

$$\delta_{\omega^\gamma} \in P_{\gamma \cdot [1/\varepsilon]}(\varepsilon, B_{C(\omega^\gamma \cdot n)}, B_{C(\omega^\gamma \cdot n)}^*)$$

Thus

$$\eta(\varepsilon, C(\omega^\gamma \cdot n)) \geq \gamma \cdot [1/\varepsilon] + 1.$$

This in fact is the correct order ([Samuel], others). So

$$\eta(C(\omega^\gamma \cdot k)) = \gamma \cdot \omega.$$

**Theorem 2 (Bessaga-Pelczynski).** If  $\alpha < \beta$ ,  $C(\omega^\alpha \cdot k)$  is isomorphic to  $C(\omega^\beta \cdot n)$  if and only if  $\beta < \alpha \cdot \omega$ . Consequently,  $C(\omega^{\omega^\gamma})$ ,  $\gamma < \omega_1$ , is a complete list of representatives of the isomorphism classes of  $C(K)$  for  $K$  a countable compact metric space.

If  $K$  is uncountable then  $\eta(C(K)) = \omega_1$ .

**Question 3.** Suppose that  $K$  is a compact metric space,  $X$  is a Banach space,  $T : C(K) \rightarrow X$  is bounded and  $\gamma < \omega_1$ . What is the largest ordinal  $\beta$  such that if  $\eta(\varepsilon, B_{C(K)}, T^*(B_{X^*})) \geq \omega^\gamma$ , then there is always a subspace  $Z$  of  $C(K)$  such that  $Z$  is isomorphic to  $C(\omega^{\omega^\beta})$  and  $T|_Z$  is an isomorphism?

If  $\gamma = 0$ ,  $\beta = 0$ . [Pelczynski]

If  $\gamma = 1$ ,  $\beta = 1$ . [A]

For any  $\gamma$ ,  $\omega^\beta \geq \omega \cdot \gamma$ . [Bourgain]

If  $1 \leq \zeta < \gamma < \zeta \cdot \omega$  for some  $\zeta < \omega_1$ ,  $\beta < \alpha$ .  
[A], [Gasparis]

Bourgain deduced the following result from his:

**Theorem 4 (Rosenthal).** Suppose that  $K$  is a compact metric space,  $X$  is a Banach space,  $T : C(K) \rightarrow X$  is bounded and  $T(B_{X^*})$  is non-separable, then there is a subspace  $Z$  of  $C(K)$  isomorphic to  $C([0, 1])$  such that  $T|_Z$  is an isomorphism.

A few other connections between the Szlenk index and the spaces of continuous function are the following:

**Theorem 5 (A-Benyamini).** If  $X$  is a separable  $\mathcal{L}_\infty$ -space,  $\varepsilon > 0$ , and  $\eta(\varepsilon, X) \geq \omega^\gamma$ , then  $C(\omega^{\omega^\gamma})$  is isomorphic to a quotient of  $X$ .

A major step in the solution of the separable injective problem was

**Theorem 6.** (Zippin's Lemma) Let  $X$  be a Banach space with separable dual,  $\varepsilon > 0$  and let  $F$  be a  $w^*$ -totally disconnected subset of  $B_{X^*}$  which is  $(1 - \varepsilon)$ -norming. Then there is a countable ordinal  $\alpha < \omega^{\eta(\varepsilon/8, X)+1}$ , a subspace  $Y$  of  $C(F)$ , isometric to  $C(\alpha)$  such that for every  $x \in X$  there is a  $y \in Y$  with  $\|\hat{x}|_F - y\| \leq \varepsilon(1 - \varepsilon)^{-1} \|\hat{x}|_F\|$ .

In Szlenk's definition of the index the weakly null sequences play an important role. If we think of a Banach space  $X$  as a subspace of  $C(B_{X^*}, w^*)$ , then we are looking at sequences of functions which are converging pointwise to 0, but not uniformly and the Szlenk sets are measuring the non-uniform convergence.

Let us change our focus to the behavior of a single weakly null not norm null sequence  $(f_n)$ . A classical theorem of Mazur asserts that there is a sequence  $(g_k)$  such that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|g_k\| &= 0, \\ g_k &= \sum_{n \in F_k} a_n f_n, \quad \text{where } a_n > 0 \text{ for all } n, \\ 1 &= \sum_{n \in F_k} a_n, \text{ for all } k, \\ F_1 &< F_2 < \dots < F_k < F_{k+1} < \dots \\ (F_k &< F_{k+1} \text{ iff } \max F_k < \min F_{k+1}). \end{aligned}$$

This result is very much an existence statement.

The simplest possible convex combination is an average. Consider the following:

$$\mathcal{S}_1 = \{F \subset \mathbb{N} : |F| \leq F\} \cup \{\emptyset\}$$

It is easy to see that  $\mathcal{S}_1$  is closed in  $2^{\mathbb{N}}$  and thus compact. Moreover  $\mathcal{S}_1^{(\omega)} = \{\emptyset\}$ . Define a sequence of functions on  $\mathcal{S}_1$  by

$$f_n = 1_{\{F: n \in F\}} \text{ for all } n \in \mathbb{N}.$$

If we think of the sequence  $(f_n)$  as a sequence in  $C(\mathcal{S}_1)$ , it converges pointwise to 0. However if  $G$  is any finite subset of  $\mathbb{N}$

$$\sum_{n \in G} f_n(F) \geq \frac{|G|}{2}$$

where  $F = \{m \in G : m \geq (|G| + 1)/2\}$ . Hence

$$\left\| \frac{1}{|G|} \sum_{n \in G} f_n \right\| \geq \frac{1}{2}.$$

Thus no sequence of averages of elements from  $(f_n)$  converges to 0 in norm.

This example is essentially that given by Schreier in 1930. For each ordinal  $\alpha < \omega_1$  there is a family of subsets of  $\mathbb{N}$ ,  $\mathcal{S}_\alpha$ , which has analogous properties, [A-Argyros],[A-Odell].

Let  $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$  and suppose that  $\mathcal{S}_\alpha$  has been defined. Let

$$\mathcal{S}_{\alpha+1} = \left\{ \bigcup_{i=1}^k F_i : k \leq F_1 < F_2 < \cdots < F_k, \right. \\ \left. F_i \in \mathcal{S}_\alpha, 1 \leq i \leq k, k \in \mathbb{N} \right\} \cup \{\emptyset\}.$$

If  $\alpha$  is a limit ordinal, let  $(\alpha_n)$  be a strictly increasing sequence of ordinals with limit  $\alpha$  and define

$$\mathcal{S}_\alpha = \bigcup_{n=1}^{\infty} \{F \in \mathcal{S}_{\alpha_n} : n \leq F\} \cup \{\emptyset\}.$$

The families  $\mathcal{S}_\alpha$  have the following properties:

- If  $F = \{m_i : 1 \leq i \leq j\}$ ,  $(m_i)_{i=1}^j$  increasing, then any increasing sequence  $(n_i)_{i=1}^j$  with  $m_i \leq n_i$  for all  $i$ ,  $\{n_i : 1 \leq i \leq j\} \in \mathcal{S}_\alpha$ . (spreading)
- If  $F \in \mathcal{S}_\alpha$  and  $G \subset F$ , then  $G \in \mathcal{S}_\alpha$ . (hereditary)
- $\mathcal{S}_\alpha$  is homeomorphic to  $[1, \omega^{\omega^\alpha}]$  in  $2^\mathbb{N}$ .
- The sequence of functions on  $\mathcal{S}_\alpha$  defined by  $f_n^\alpha = 1_{\{F: n \in F\}}$  for  $n \in \mathbb{N}$  converges to 0 pointwise.

Sometimes it is useful to insert some additional families of sets between  $\mathcal{S}_\alpha$  and  $\mathcal{S}_{\alpha+1}$ , e.g., by taking a fixed natural number  $n$  and  $k \leq n$  in the inductive definition. This makes some induction arguments easier. (Farmaki has formalized this.)

Some work has been done on combinatorial and permanence properties of the Schreier families. For example, if we have two families of finite subsets of  $\mathbb{N}$ ,  $\mathcal{F}$  and  $\mathcal{G}$ , with properties like those of the Schreier families (hereditary, spreading and closed ) it is possible to compose the families:

$$\mathcal{G}[\mathcal{F}] = \left\{ \bigcup_{i=1}^k F_i : F_1 < F_2 < \cdots < F_k, F_i \in \mathcal{F} \right. \\ \left. \text{for } i \leq k, (\min F_i)_{i=1}^k \in \mathcal{G} \right\}.$$

Using this operation  $\mathcal{S}_{\alpha+1} = \mathcal{S}_1[\mathcal{S}_\alpha]$ .

Moreover it was shown [Odell–Tomczak-Jaegermann–Wagner] that the Schreier classes are almost closed under this composition operation, i.e., for  $\alpha$  and  $\beta$  there are infinite subsets  $M, N$  of  $\mathbb{N}$  such that

$$\mathcal{S}_\alpha[\mathcal{S}_\beta](M) \subset \mathcal{S}_{\beta+\alpha} \text{ and } \mathcal{S}_{\beta+\alpha}(N) \subset \mathcal{S}_\alpha[\mathcal{S}_\beta].$$

There are also some results about the relationship between hereditary families of sets and the Schreier families, e.g., [Gasparis], [Judd].

**Theorem 7.** ([G]) Let  $\mathcal{F}$  be a hereditary family of finite subsets of  $\mathbb{N}$ ,  $\alpha < \omega_1$ , and an infinite subset  $N$  of  $\mathbb{N}$ . Then there exists an infinite subset  $M$  of  $N$  so that either  $\mathcal{S}_\alpha \cap 2^M \subset \mathcal{F}$  or  $\mathcal{F} \cap 2^M \subset \mathcal{S}_\alpha$ .

The Schreier families give a hierarchy that has proved quite useful for analyzing things that depend on “blocking”. In particular various averaging schemes have made use of the families.

These can be found in papers investigating more constructive versions of Mazur's theorem which also give some additional information, e.g., [Argyros-Merkourakis-Tsarpalias], [Argyros-Gasparis]. The Schreier families are used to make sense of repeated averaging. Thus given a sequence  $(x_n)$  one can describe constructing sequences of the form

$$\frac{\sum_{n \in G_1} x_n}{|G_1|}, \frac{\sum_{n_1 \in G_2} \frac{\sum_{n_2 \in G_{2,n_1}} x_{n_2}}{|G_{2,n_1}|}}{|G_2|}, \dots$$

The Schreier families have also been used for quantifying the  $\ell_1$  structure of an asymptotic  $\ell_1$ -space, [O-T-W]. Tsirelson space (the dual of the original example) has the property that a sequence of  $n$  blocks of the basis starting after  $n$  are equivalent to the  $\ell_1^n$  basis. These blocks have supports in  $\mathcal{S}_1$ . In [O-T-W] parameters are introduced to give useful answers to questions about Tsirelson space and its generalizations such as: What sequences of blocks with supports in  $\mathcal{S}_\alpha$  are equivalent to the  $\ell_1^n$ -basis? What if we pass to a block of the original basis and then take further blocks in some  $\mathcal{S}_\alpha$ ?

There is also an interesting family of Banach sequence spaces defined using the Schreier sets. The  $\alpha$ th Schreier space,  $X_\alpha$ , has norm

$$\|(x_n)\|_\alpha = \sup_{F \in \mathcal{S}_\alpha} \left| \sum_{n \in F} x_n \right|.$$

The natural basis  $(f_n^\alpha)$  is unconditional. In [Gasparis-Leung] it is shown that the complemented subspace structure of these spaces is very rich. Because these spaces seem analogous to  $C(\omega^{\omega^\alpha})$  but with unconditional basis one can start asking questions similar to those that have been investigated for the spaces  $C(\omega^{\omega^\alpha})$ .

**Question 8.** Can subspaces of  $C(K)$  which have a subspace isomorphic to  $X_\alpha$  be detected by some ordinal index? If a subset of  $B_{C(K)}$  norms a subspace  $Y$  which is isomorphic to  $X_\alpha$  must it also norm a subspace isomorphic to  $C(\omega^{\omega^\alpha})$ ?

A recurrent theme in much of this work is a need to deal with  $\ell_1$  or  $\ell_1^+$  behavior but in spaces that do not contain  $\ell_1$ . Bourgain introduced a  $\ell_1$  index in an attempt to describe “partial containment of  $\ell_1$ .” For this notion we need to recall a standard method of defining an ordinal index on well-founded trees. Our trees will be constructed by taking finite sequences from a fixed set and ordering them by extension. The assumption of well-foundedness means that there is no infinite linearly ordered subset (branch).

Now we can define a notion of derived set by stripping off terminal elements.

Let  $T$  be a tree on  $X$  and  $T^0 = T$ . If  $T^\alpha$  is defined, let

$$T^{\alpha+1} = \{(x_i)_{i=1}^k \in T^\alpha : \exists y_j \in X, j = 1, 2, \dots, n, \text{ for some } n \in \mathbb{N}, (x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_n) \in T^\alpha\}$$

For limit ordinals  $T^\beta = \bigcap_{\alpha < \beta} T^\alpha$ . The order of the tree, is the smallest ordinal  $\alpha$  such that  $T^\alpha = \emptyset$ .

Fix a constant  $K \geq 1$ . Let  $X$  be a Banach space and let

$$T = \{(x_i)_{i=1}^k : x_i \in X, \|x_i\| = 1, \text{ for } i = 1, 2, \dots, k, k \in \mathbb{N},$$

and

$$K^{-1} \sum_{i=1}^k |a_i| \leq \left\| \sum_{i=1}^k a_i x_i \right\| \leq \sum_{i=1}^k |a_i|$$

for all  $a_i \in \mathbb{R}, 1 \leq i \leq k\}$

This is Bourgain's local  $\ell_1$ -index for constant  $K$ . We take the supremum over all  $K \geq 1$  to get an index for the space.

- The tree  $T$  is well-founded exactly when there is no normalized sequence in  $X$  which is  $K$ -equivalent to the usual unit vector basis of  $\ell_1$ .
- There are reflexive spaces with large  $\ell_1$  index.
- The index of the Schreier space  $X_\alpha$  is  $\omega^{\alpha+1}$ . (A-Judd-Odell)

Because of the results of James on the non-distortability of  $\ell_1$ , it might be expected that the constant  $K$  plays a minimal role in this index. This has been confirmed in [Judd-Odell] where it is shown that there is a method of constructing trees with improved constants  $K$  with controlled decrease in index.

If we remove the absolute values in the definition above we get a local  $\ell_1^+$  index.

$$T = \{(x_i)_{i=1}^k : x_i \in X, \|x_i\| = 1, \text{ for } i = 1, 2, \dots, k, \\ k \in \mathbb{N}, \text{ and } K^{-1} \sum_{i=1}^k a_i \leq \left\| \sum_{i=1}^k a_i x_i \right\| \\ \text{for all } a_i \in \mathbb{R}^+, 1 \leq i \leq k\}$$

The  $\ell_1^+$  index is countable exactly when the space is reflexive. ([James], [Milman-Milman])  
The  $\ell_1^+$  condition is closely related to Rosenthal's notion of a wide-(s) sequence.

Sometimes it is possible to create a Szlenk-like index which gives similar information to that of an index defined by trees.

Suppose that  $(f_n)$  is a bounded sequence of continuous functions on a compact metric space  $U$  which converges pointwise. Fix  $\varepsilon > 0$  and define for each  $n, m \in \mathbb{N}$ ,

$$A_{n,m}^+ = \{u \in U : f_n(u) - f_m(u) > \varepsilon\}$$

$$A_{n,m}^- = \{u \in U : f_n(u) - f_m(u) < -\varepsilon\}$$

Now we define  $O^0(\varepsilon, (f_n), U) = U$  and for all  $\alpha < \omega_1$ ,

$$\mathcal{O}^{\alpha+1}(\varepsilon, (f_n), U) = \{u \in \mathcal{O}^\alpha(\varepsilon, (f_n), U) :$$

$$\forall V \text{ open, } u \in V, \exists N \in \mathbb{N} \ni \forall n \geq N, \exists M \in \mathbb{N} \ni$$

$$\cap_{m \geq M} A_{n,m}^+ \cap \mathcal{O}^\alpha(\varepsilon, (f_n), U) \cap V \neq \emptyset$$

or

$$\cap_{m \geq M} A_{n,m}^- \cap \mathcal{O}^\alpha(\varepsilon, (f_n), U) \cap V \neq \emptyset\}.$$

As usual we take the intersection at a limit ordinal. The corresponding index is the  $\varepsilon$  oscillation index of the sequence  $(f_n)$ .

The condition imposed on the points in the underlying topological space  $U$  in the definition of the oscillation sets is much more restrictive than that in the definition of the Szlenk sets and thus this index is smaller than the Szlenk index. If we take the Schreier family,  $\mathcal{S}_\alpha$  as the space  $U$  and the associated sequence  $(f_n^\alpha)$ , then for  $0 < \varepsilon < 1$ ,  $\mathcal{O}^{\omega^\alpha}(\varepsilon, (f_n^\alpha), U) \neq \emptyset$ . This shows that the oscillation index is maximal for this sequence and that in this case the Szlenk index can be computed by using a single weakly null sequence.

In [A-A] it is shown that the oscillation index gives information about the local  $\ell_1$  index.

**Theorem 9.** If  $(f_n)$  is a bounded, pointwise converging sequence on a compact metric space  $U$  and for some  $\varepsilon > 0$  and  $\alpha$ ,  $\mathcal{O}^\alpha(\varepsilon, (f_n), U) \neq \emptyset$ , then there is a local  $\ell_1$ -index tree on  $(f_n)$  with index at least  $\alpha/2$ .

Bourgain shows that the local  $\ell_1$ -index is related to the Lavrentiev index of the pointwise limit of a sequence of continuous functions. The oscillation index of the sequence is also essentially bounded below by the Lavrentiev index of the limit.

We want to consider one more index which is based on trees. We use  $\ell_1^+$  trees but we make an additional requirement that if

$$(x_1, x_2, \dots, x_n, y_j) \in T \quad \text{for } j = 1, 2, \dots$$

then

$$y_j \xrightarrow{w} 0.$$

We also require that "forking" must be infinite, i.e., if a sequence in the tree is not maximal then there must be infinitely many extensions of the same length. (Some additional technical assumptions are also imposed.) To make sure that we can actually have interesting trees of this type, we need many weakly null sequences. Thus separability of the dual or at least not having  $\ell_1$  in the space is generally assumed.

Once we have suitably restricted the trees which are allowed we compute the order of each tree as before and take the supremum over all  $K \geq 1$ . The resulting index is called the weakly null (local)  $\ell_1^+$  index.

**Theorem 10.** ([A-J-O]) If  $X$  is a separable Banach space not containing  $\ell_1$  then the Szlenk index and the weakly null  $\ell_1^+$  index are equal.

Here we have an example of two indices of a Banach space which superficially appear to give quite different information and are certainly computed in very different ways, yet yield the same index.

**Question 11.** What other ordinal indices have “dual” versions?

One way that the  $\varepsilon$  Szlenk derived sets have been used is to carve the unit ball of the dual or other subset of the dual into pieces of norm diameter at most  $\varepsilon$  as in Zippin's Lemma. This is adequate for some purposes but other times it would be convenient to use more than one  $\varepsilon$  or even a sequence  $(\varepsilon_k)$  which decreases to zero. Unfortunately the Szlenk sets for different values of  $\varepsilon$  are only loosely related. Recently, I have been working with an iterated version of the Szlenk sets. Here is the basic idea.

Let  $(\varepsilon_j)_{j=1}^{\infty}$  be a decreasing sequence of positive numbers and let  $A \subset X$  and  $B \subset X^*$ .

For a sequence of length one define

$$P_{\alpha}((\varepsilon_1), A, B) = P_{\alpha}(\varepsilon_1, A, B)$$

for all  $\alpha$ .

Suppose that

$$P_\gamma((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{j-1}), A, B)$$

has been defined for all countable ordinals  $\gamma$ .  
Let

$$P_0((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j), A, B) = B.$$

Suppose that  $P_\alpha((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j), A, B)$  has been defined and that  $\gamma$  is an ordinal such that

$$\begin{aligned} P_\gamma((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{j-1}), A, B) \\ \supset P_\alpha((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j), A, B) \\ \not\supseteq P_{\gamma+1}((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{j-1}), A, B). \end{aligned}$$

Define

$$\begin{aligned} P_{\alpha+1}((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j), A, B) = \\ P_1(\varepsilon_j, A, P_\alpha((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j), A, B)) \\ \cup P_{\gamma+1}((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{j-1}), A, B). \end{aligned}$$

If  $\beta$  is a limit ordinal define

$$\begin{aligned} P_\beta((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j), A, B) \\ = \cap_{\alpha < \beta} P_\alpha((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j), A, B). \end{aligned}$$

The point here is that the derived sets for  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j)$  are slipped in between the derived sets for  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{j-1})$  so as to refine the differences

$$P_\gamma((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{j-1}), A, B) \setminus P_{\gamma+1}((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{j-1}), A, B).$$

Using this idea and a further modification of the basic Szlenk index, it is possible to prove a generalization of Zippin's Lemma which allows for refinement.