12. Normalization of intertwining operators.

Let π be a generic irreducible admissible representation of $\mathbf{M}(F)$, F local field. Let $A(s,\pi,w_0)$ be the intertwining operator for $I(s,\pi)$. We normalize it in the following way. Let

$$N(s,\pi,w_0)=\prod_{i=1}^{m}\frac{L(1+is,\pi,r_i)\epsilon(is,\pi,r_i,\psi)}{L(is,\pi,r_i)}A(s,\pi,w_0). \tag{12.1}$$

Conjecture 12.1. *$N(s,\pi,w_0)$ is holomorphic and non-vanishing for $Re(s)\geq\frac{1}{2}$.*

12.1 π is supercuspidal.

Proposition 12.2. *$N(s,\pi,w_0)$ is holomorphic and non-vanishing except possibly at $Re(s)=-1$ or $-\frac{1}{2}$.*

Proof. Write

$$N(s,\pi,w_0)=\prod_{i=1}^{m}L(1+is,\pi,r_i)\epsilon(is,\pi,r_i,\psi)\frac{A(s,\pi,w_0)}{\prod_{i=1}^{m}L(is,\pi,r_i)}.$$

Here $\prod_{i=1}^{m}L(1+is,\pi,r_i)$ has a pole possibly at $Re(s)=-1$ or $-\frac{1}{2}$, and $\frac{A(s,\pi,w_0)}{\prod_{i=1}^{m}L(is,\pi,r_i)}$ is entire and non-vanishing. □

12.2 π is tempered, generic.

Proposition 12.3. *$N(s,\pi,w_0)$ is holomorphic and non-vanishing for $Re(s)\geq 0$, except the 4 cases where Shahidi's conjecture (Conjecture 11.13) is not proved.*

Proof. By Harish-Chandra, $A(s,\pi,w_0)$ is holomorphic and non-vanishing for $Re(s)>0$. By Shahidi's conjecture, $L(s,\pi,r_i)$ is holomorphic for $Re(s)>0$. Hence our assertion follows. For $Re(s)=0$, it is a result of the theory of R-groups. □

Proposition 12.4. *Let σ,τ be tempered representations of GL_k,GL_l, resp. Then the normalized operator $N(s,\sigma\otimes\tau,w_0)$ is holomorphic and non-vanishing for $Re(s)>-1$.*

12.3 π is non-tempered, generic.

Let π be a generic irreducible, non-tempered representation of $\mathbf{G}(F)$, F local. By Langlands' classification theorem, $\pi=J(\Lambda_0,\sigma)$, where σ is an irreducible generic tempered representation of $\mathbf{M}(F)$, and Λ_0 is in the positive Weyl chamber of $\mathfrak{a}_{\mathbb{C}}^*$. We call $I(\Lambda_0,\sigma)$ the standard module.

Standard module conjecture. *$\pi=I(\Lambda_0,\sigma)$, i.e., $I(\Lambda_0,\sigma)$ is irreducible.*

Assume the standard module conjecture. Then

$$I(s,\pi)=Ind_{P_1}^{G}\,\sigma\otimes exp(\langle s\tilde{\alpha}+\Lambda_0,H_{P_1}(\,)\rangle),\quad A(s,\pi,w_0)=A(s\tilde{+}\Lambda_0,\sigma,w_0).$$

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Lemma 12.5. *If $s\tilde{\alpha}+\Lambda_0$ is in the corresponding positive Weyl chamber for $Re(s) \geq \frac{1}{2}$ and Shahidi's conjecture (Conjecture 11.13) is true for each rank-one situation, then $N(s, \pi, w_0)$ is holomorphic for $Re(s) \geq \frac{1}{2}$.*

Lemma 12.6. *Assume Conjecture 11.13 for each rank-one situation. If $N(s\tilde{\alpha} + \Lambda_0, \sigma, w_0)$ is holomorphic at $s = s_0$, then it is non-zero at $s = s_0$.*

We need a result on bounds of Fourier coefficients.

Proposition 12.7. *Let $\pi = \otimes_v \pi_v$ be a unitary, generic cuspidal representation of $\mathbf{M}(\mathbb{A})$. Fix a place v where π_v is non-tempered. Assume the standard module conjecture and write π_v as $\pi_v = I_{M_0}(\Lambda_0, \pi_0)$. Assume Conjecture 11.13 for each rank-one situation so that Lemma 12.5 may be applied. Then the normalized operator $N(s, \pi_v, w_0)$ and the local L-function $L(s, \pi_v, r_1)$ are holomorphic for $Re(s) \geq 1$.*

Proof. Fix a place w where π_w is spherical. By checking the L-functions in section 6 (or use [Ki-Sh, Proposition 2.1]), we can take a grössencharacter χ such that

(1) $\chi_v = 1$ and χ_w is highly ramified;
(2) $w_0(\pi \otimes \chi) \not\simeq \pi \otimes \chi$;
(3) $w_0'(\pi_i' \otimes \chi) \not\simeq \pi_i' \otimes \chi$ for all i, where π_i' is as in Theorem 7.1, namely, $L(s, \pi, r_i) = L(s, \pi_i', r_1')$, and w_0' is the Weyl group element for π_i'.

Then $M(s, \pi \otimes \chi, w_0)$ and $M(s, \pi_i' \otimes \chi, w_0')$ are holomorphic for $Re(s) \geq 0$ by Corollary 5.5. Hence by omitting χ, we can assume that $M(s, \pi, w_0)$ and $M(s, \pi_i', w_0')$ are holomorphic for $Re(s) \geq 0$.

Recall (See [Sh3, (2.7)])

$$M(s, \pi, w_0)f = \prod_{i=1}^{m} \frac{L_S(is, \pi, r_i)}{L_S(1 + is, \pi, r_i)} \otimes_{u \notin S} \tilde{f}_u \otimes \otimes_{u \in S} A(s, \pi_u, w_0) f_u, \tag{12.2}$$

where S is a finite set of places including archimedean places such that $v \in S$ and π_u is unramified for $u \notin S$, and $f = \otimes_u f_u$ is such that for each $u \notin S$, f_u is the unique K_u-fixed function normalized by $f_u(e_u) = 1$ and $\tilde{f}_u$ is the K_u-fixed function in the space of $I(-s, w_0(\pi_u))$, normalized in the same way.

Now by induction, we show that for all i, $L_S(s, \pi, r_i)$ is holomorphic for $Re(s) \geq \frac{1}{2}$, and has no zeros for $Re(s) \geq 1$. For each $u \in S$, $A(s, \pi_u, w_0)$ is not a zero operator. Since $M(s, \pi, w_0)$ is holomorphic for $Re(s) \geq 0$, the quotient $\prod_{i=1}^m \frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}$ is holomorphic for $Re(s) \geq 0$. Now starting at $Re(s) > N_0$, where $\prod_{i=1}^m L_S(is, \pi, r_i)$ is absolutely convergent, and arguing inductively, we can see that $\prod_{i=1}^m L_S(is, \pi, r_i)$ is holomorphic for $Re(s) \geq 0$.

Next, recall from Corollary 9.5 that $\prod_{i=1}^m L_S(1+is, \pi, r_i)$ has no zeros for $Re(s) \geq 0$.

Now we apply the induction on m: First, let $m = 1$. It is clear. Suppose our assertion is true for $L_S(s, \pi, r_i)$, $i = 2, ..., m$, i.e., for all $2 \leq i \leq m$, $L_S(s, \pi, r_i)$ is holomorphic for $Re(s) \geq \frac{1}{2}$, and has no zeros for $Re(s) \geq 1$. Since $\prod_{i=1}^m L_S(is, \pi, r_i)$ is holomorphic for $Re(s) \geq 0$, $L_S(s, \pi, r_1)$ is holomorphic for $Re(s) \geq \frac{1}{2}$. Since

$\prod_{i=1}^m L_S(1+is,\pi,r_i)$ has no zeros for $Re(s)\geq 0$, $L_S(s,\pi,r_1)$ has no zeros for $Re(s)\geq 1$. This finishes the induction step.

Applying the induction again on m, this time for the local L-functions, we can assume that $L(s,\pi_v,r_i)$, $i=2,...,m$, is holomorphic for $Re(s)\geq 1$. Now we normalize $A(s,\pi_v,w_0)$ as in (12.1). Since for each $u\in S$, $u\neq v$, $A(s,\pi_u,w_0)$ is not a zero operator, pick f_u, $u\in S$, $u\neq v$, so that $A(s,\pi_u,w_0)f_u\neq 0$. Then (12.2) is written as

$$M(s,\pi,w_0)f=\prod_{i=1}^m \frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}\prod_{i=1}^m \frac{L(is,\pi_v,r_i)}{L(1+is,\pi_v,r_i)}\otimes_{u\notin S}\tilde{f}_u\otimes$$
$$\otimes_{u\in S,u\neq v}A(s,\pi_u,w_0)f_u\otimes\frac{N(s,\pi_v,w_0)}{\prod_{i=1}^m \epsilon(s,\pi_v,r_i,\psi_v)}.$$

Now pick $N_0\geq 1$ so large that $L(1+s,\pi_v,r_1)$ has no poles for $Re(s)\geq N_0$. If $Re(s)\geq N_0-1$, the left hand side is holomorphic and each term on the right hand side except possibly $N(s,\pi_v,w_0)$ is not zero there. Hence the normalized operator $N(s,\pi_v,w_0)$ is holomorphic for $Re(s)\geq N_0-1$. By Lemma 12.6, $N(s,\pi_v,w_0)$ is non-vanishing for $Re(s)\geq N_0-1$ (Apply it by identifying $N(s,\pi_v,w_0)$ with $N(\Lambda,\pi_0,w_0)$). Hence $L(s,\pi_v,r_1)$ has no poles for $Re(s)\geq N_0-1$. Arguing inductively, we see that $L(s,\pi_v,r_1)$ has no poles for $Re(s)\geq 1$. □

The above proposition has a very important application when applied to E_8-2 case. Let $\pi=\otimes_v\pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$. Let $diag(\alpha_v,\beta_v)$ is the Satake parameter for an unramified π_v. Let $\pi_1=Sym^3(\pi)\otimes\omega_\pi^{-1}$, constructed in [Ki-Sh], and $\pi_2=Sym^4(\pi)$, constructed in [Ki5]. Then we obtain the L-function $L(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5)$ in E_8-2 case. Let S be a finite set of places of finite places such that π_v is unramified for $v\notin S$, $v<\infty$. By standard calculation, we have

$$L_S(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5)=L_S(s,\pi,Sym^9)$$
$$L_S(s,\pi,Sym^7\otimes\omega_\pi)L_S(s,\pi,Sym^5\otimes\omega_\pi^2)^2L_S(s,Sym^3(\pi)\otimes\omega_\pi^3)^2L_S(s,\pi\otimes\omega_\pi^4).$$

This immediately implies meromorphic continuation and the functional equation of the 9th symmetric power L-functions. Now Proposition 12.7 implies that for each v, $L(s,\pi_{1v}\otimes\pi_{2v},\rho_4\otimes\wedge^2\rho_5)$ is holomorphic for $Re(s)\geq 1$, and so is $L(s,\pi_v,Sym^9)$. Therefore we have

Corollary 12.8. *Let $\pi=\otimes_v\pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$. Let π_v be a local (finite or infinite) spherical component, given by $\pi_v=Ind(|\ |_v^{s_{1v}}\otimes|\ |_v^{s_{2v}})$. Then*

$$|Re(s_{iv})|<\frac{1}{9}.$$

If $F=\mathbb{Q}$, $v=\infty$, this means

$$\lambda_1=\frac{1}{4}(1-s^2)>\frac{77}{324}\approx 0.238,$$

where $s=2s_{1v}=-2s_{2v}$ and λ_1 is the first eigenvalue of the Laplace operator on the corresponding hyperbolic space.

We apply Proposition 12.7 to $\mathbf{G}_n=Sp(2n),SO(2n+1),SO(2n)$.

Proposition 12.9. *Let $\tau = \otimes_v \tau_v$ be a generic cuspidal representation of $\mathbf{G}_n(\mathbb{A})$. Let τ_v be a non-tempered component and write*

$$\tau_v = Ind\,|det|^{\beta_1}\tau_1 \otimes \cdots \otimes |det|^{\beta_k}\tau_k \otimes \tau_0,$$

(This is the standard module conjecture, proved by Muić [Mu3]) where $\tau_1, ..., \tau_k$ are discrete series representations of $GL_{n_i}(F_v)$ and τ_0 is a generic tempered representation of $\mathbf{G}_r(F_v)$, and $0 < \beta_k \leq \cdots \leq \beta_1$. Then $\beta_1 < 1$.

We need a lemma.

Lemma 12.10 (Rogawski [Ro]). *Let $\sigma_1, ..., \sigma_k$ be discrete series representations of $GL_n(F_{v_i})$. Then there exists a cuspidal representation $\pi = \otimes_v \pi_v$ such that $\pi_{v_i} \simeq \sigma_i$ for each i.*

Proof of Proposition 12.9. Let $\sigma = \otimes_v \sigma_v$ be a cuspidal represenation of $GL_{n_1}(\mathbb{A})$ such that $\sigma_v \simeq \tilde{\tau}_1$. Then by Proposition 12.7, $L(s, \sigma_v \times \tau_v)$ is holomorphic for $Re(s) \geq 1$.

Note that

$$I(s, \sigma_v \otimes \tau_v) = Ind\,|det|^s\tilde{\tau}_1 \otimes |det|^{\beta_1}\tau_1 \otimes \cdots \otimes |det|^{\beta_k}\tau_k \otimes \tau_0.$$

Hence

$$L(s, \sigma_v \times \tau_v) = L(s-\beta_1, \tilde{\tau}_1 \times \tau_1)L(s, \tilde{\tau}_1 \times \tau_0)\prod_{i=1}^{k} L(s-\beta_i, \tilde{\tau}_1 \times \tau_i)\prod_{i=1}^{k} L(s+\beta_i, \tilde{\tau}_1 \times \tilde{\tau}_i).$$

Here $L(s - \beta_1, \tilde{\tau}_1 \times \tau_1)$ has a pole at $s = \beta_1$. Therefore $\beta_1 < 1$. $\square$

We have

Theorem 12.11. *Conjecture 12.1 holds except possibly for the following 10 cases; (xxx) ($E_6 \subset E_7$), $E_8 - 4$ and (xxxii) ($E_7 \subset E_8$) (These 3 cases are where standard module conjecture is not available); (xviii) ($B_3 \subset F_4$), (xxii) ($C_3 \subset F_4$), (xxiv) ($D_5 \subset E_6$), $E_7 - 3$, (xxvi) ($D_6 \subset E_7$), $E_8 - 3$, and (xxviii) ($D_7 \subset E_8$) (These 7 cases are where the Levi subgroup contains a group of type B_3, C_3, D_n).*

By Proposition 12.3, we only have to show for non-tempered π_v. Using standard module conjecture, we denote

$$I(s, \pi_v) = I(s\tilde{\alpha} + \Lambda_0', \sigma_v') \subset I(s\tilde{\alpha} + \Lambda_0, \sigma_v),$$

where σ_v' is a generic tempered representation and σ_v is a generic discrete series. In what follows, we can assume that s is real. All we need to do is that for $\frac{1}{2} \leq s < 1$, rank-one normalized operators are holomorphic. We can see by checking case by case that in the cases under consideration, rank-one operators for the exceptional 4 cases which were excluded in Theorem 11.14 do not appear. By identifying roots of $\mathbf{G}$ with respect to a parabolic subgroup, with those of $\mathbf{G}$ with respect to the maximal torus, it is enough to check $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee \rangle > -1$ if the rank-one operators are for those of $GL_k \times GL_l \subset GL_{k+l}$. If there are rank-one operators for other situation, we need to check $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee \rangle > -\frac{1}{2m}$. We check case by case.

We illustrate the proof in the case of $GL_k \times SO_{2l+1} \subset SO_{2n+1}$. Let σ, τ be generic, non-tempered, unitary representations of $GL_k(F)$, $SO_{2l+1}(F)$, resp. where F is a local field. Recall the classification of unitary representations of $GL_n(F)$ due to Tadic [Ta]: A generic, unitary representation σ is of the form

$$\sigma = Ind\,|det|^{r_1}\sigma_1 \otimes \cdots \otimes |det|^{r_p}\sigma_p \otimes \sigma_{p+1} \otimes |det|^{-r_p}\sigma_p \otimes \cdots \otimes |det|^{-r_1}\sigma_1,$$

where $0 < r_p < \cdots < r_1 < \frac{1}{2}$ and $\sigma_1, ..., \sigma_p, \sigma_{p+1}$ are tempered representations of $GL_{n_i}(F)$. By Proposition 12.9,

$$\tau = Ind\,|det|^{\beta_1}\tau_1 \otimes \cdots \otimes |det|^{\beta_q}\tau_q \otimes \tau_0,$$

where $\tau_1, ...\tau_q$ are tempered representations of $GL_{m_j}(F)$, τ_0 is a generic tempered representation of $SO_{2r+1}(F)$, and $0 < \beta_q < \cdots < \beta_1 < 1$. Then

$$\begin{aligned} I(s, \sigma \otimes \tau) = Ind\,|det|^s\sigma \otimes \tau = Ind\,|det|^{s+r_1}\sigma_1 \otimes \cdots \otimes |det|^{s+r_p}\sigma_p \otimes |det|^s\sigma_{p+1} \otimes \\ |det|^{s-r_p}\sigma_p \otimes \cdots \otimes |det|^{s-r_1}\sigma_1 \otimes |det|^{\beta_1}\tau_1 \otimes \cdots \otimes |det|^{\beta_q}\tau_q \otimes \tau_0. \end{aligned}$$

Here $N(s, \sigma \otimes \tau, w_0)$ is a product of rank-one normalized operators $N(s \pm r_i \pm \beta_j, \sigma_i \otimes \tau_j)$ and $N(s \pm r_i, \sigma_i \otimes \tau_0)$. Each of the operators are holomorphic and non-vanishing by Proposition 12.3 and 12.4, since $Re(s - r_1 - \beta_1) > -1$ (the worse case) if $Re(s) \geq \frac{1}{2}$.

12.4 Application to reducibility criterion. Recall the following.

Theorem 12.12. *Let F be p-adic. Suppose π is a unitary supercuspidal representation of $\mathbf{M}(F)$ such that $w_0(\pi) \simeq \pi$. Then there exists a nonnegative number s_0 such that $I(s, \pi)$ is reducible at $s = \pm s_0$, and irreducible everywhere else.*

If π is generic, then $s_0 \in \{0, \frac{1}{2}, 1\}$.

Theorem 12.13. *Let π be a generic discrete series representation of $\mathbf{M}(F)$. Then $I(s, \pi)$ is irreducible at $s = 0$ if and only if there exists a unique i, $1 \leq i \leq m$ such that $L(s, \pi, r_i)$ has a pole at $s = 0$. The pole is simple, i.e, $\prod_{i=1}^m L(is, \pi, r_i)$ has a simple pole at $s = 0$.*

Theorem 12.14. *Let π be a generic tempered representation of $\mathbf{M}(F)$ such that the standard module conjecture is true, namely, if $J(s, \pi)$ is generic, then $J(s, \pi) = I(s, \pi)$ for $Re(s) > 0$. Then for $Re(s) > 0$, $I(s, \pi)$ is irreducible if and only if $\prod_{i=1}^m L(1 - is, \pi, r_i)^{-1} \neq 0$.*

13. Holomorphy and bounded in vertical strips.

13.1 Holomorphy of L-functions. We prove

Proposition 13.1. *Let $\pi = \otimes_v \pi_v$ be a unitary, generic cuspidal representation of $\mathbf{M}(\mathbb{A})$. Assume Conjecture 12.1. Then there exists a grössencharacter χ such that for all i, $L(s, \pi \otimes \chi, r_i)$ is entire, and $L(s, \pi \otimes \chi, r_i)$ has no zeros for $Re(s) \geq 1$.*

Proof. Fix a place w where π_w is spherical. By checking the L-functions in section 6 (or use [Ki-Sh, Proposition 2.1]), we can take a grössencharacter χ (by Grunwald-Wang theorem) such that

(1) χ_w is highly ramified;
(2) $w_0(\pi \otimes \chi) \not\simeq \pi \otimes \chi$;
(3) $w_0'(\pi_i' \otimes \chi) \not\simeq \pi_i' \otimes \chi$ for all i, where π_i' is as in Theorem 7.1, namely, $L(s, \pi, r_i) = L(s, \pi_i', r_1')$, and w_0' is the Weyl group element for π_i'.

Then $M(s, \pi \otimes \chi, w_0)$ and $M(s, \pi_i' \otimes \chi, w_0')$ are holomorphic for $Re(s) \geq 0$ by Corollary 5.5. Hence by omitting χ, we can assume that $M(s, \pi, w_0)$ and $M(s, \pi_i', w_0')$ are holomorphic for $Re(s) \geq 0$. Then

$$M(s, \pi, w_0) f = \prod_{i=1}^{m} \frac{L(is, \pi, r_i)}{L(1 + is, \pi, r_i)\epsilon(is, \pi, r_i)} \otimes N(s, \pi_v, w_0) f_v.$$

By assumption, $N(s, \pi_v, w_0)$ is holomorphic and non-zero for $Re(s) \geq \frac{1}{2}$. Then

$$\prod_{i=1}^{m} \frac{L(is, \pi, r_i)}{L(1 + is, \pi, r_i)}$$

is holomorphic for $Re(s) \geq \frac{1}{2}$. Now starting at $Re(s) \geq N_0$, where $\prod_{i=1}^m L(is, \pi, r_i)$ is absolutely convergent, and arguing inductively, we can see that $\prod_{i=1}^m L(is, \pi, r_i)$ is holomorphic for $Re(s) \geq \frac{1}{2}$.

By Corollary 9.5, $\prod_{i=1}^m L(1 + is, \pi, r_i)$ has no zeros for $Re(s) \geq 0$. (This follows from the fact that the local L-functions $L(s, \pi_v, r_i)$ has no zeros at all.)

Now we use the induction on m: If $m = 1$, it is clear. Suppose it is true for $m' < m$. The induction hypothesis is that for all $i \geq 2$, $L(s, \pi, r_i)$ is holomorphic for $Re(s) \geq \frac{1}{2}$ and has no zeros for $Re(s) \geq 1$. Then clearly, $L(s, \pi, r_1)$ is holomorphic for $Re(s) \geq \frac{1}{2}$ and has no zeros for $Re(s) \geq 1$. By the functional equation, $L(s, \pi, r_i)$ is entire. □

13.2 Boundedness in vertical strips of L-functions.

In writing this section, I benefited from the conversation with M. McKee. In order to apply the converse theorem of Cogdell and Piatetski-Shapiro, it is necessary to establish the boundedness in vertical strips of automorphic L-functions. In this section, we summarize the result of Gelbart and Shahidi [Ge-Sh].

Note that $\zeta(s)$ is not bounded in vertical strips. This follows from

Proposition 13.2. *If $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $a_n \geq 0$ and $\sum \frac{a_n}{n^{\sigma_0}}$ is divergent, then $f(s)$ is not bounded in the region $\sigma > \sigma_0$, and $|t| \geq t_0 > 0$, where $s = \sigma + it$.*

But the completed L-function $\xi(s) = s(1-s)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is bounded in vertical strips. The reason is that the Γ-function controls the growth of $\zeta(s)$ as $t \to \infty$.

Lemma 13.3. $\lim_{t\to\infty} |\Gamma(\sigma+it)|e^{\frac{1}{2}\pi|t|}|t|^{\frac{1}{2}-\sigma} = \sqrt{2\pi}$. *Hence*

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}e^{-\frac{1}{2}\pi|t|}|t|^{\sigma-\frac{1}{2}}, \quad |t| \to \infty.$$

We obtain the boundedness in vertical strips by using the following theorem.

Phragmen-Lindelöf theorem. *Let $f(s)$ be holomorphic in a half-strip, $s=\sigma+it$, $a\le\sigma\le b$, $t\ge t_0\ge 0$ with fixed a,b,t_0.*

Suppose that for some $\alpha\ge 1$, we have the crude bound

$$f(\sigma+it) = O(e^{t^\alpha}), \quad t\ge t_0,$$

(This means that there exists a constant K such that $|f(\sigma+it)|\le Ke^{t^\alpha}$; we call such f function of finite order in the half-strip.) and on the sides of the half-strip we have the better bound

$$f(a+it) = O(t^M), \quad f(b+it) = O(t^M), \quad t\ge t_0.$$

Then $f(\sigma+it)=O(t^M)$ on the half-strip. In particular, if f is bounded on the sides of the half-strip, then f is bounded on the half-strip.

The following theorem is quite useful.

Theorem 13.4. *Let $f(s)$ be holomorphic in a half-strip, $s=\sigma+it$, $a\le\sigma\le b$, with fixed a,b. Suppose that f is of finite order and*

$$f(a+it) = O(|t|^{k_1}), \quad f(b+it) = O(|t|^{k_2}),$$

Then $f(\sigma+it)=O(|t|^{k(\sigma)})$ uniformly for $a\le\sigma\le b$, $k(\sigma)$ being the linear function of σ which takes the values k_1,k_2 for $\sigma=a,b$.

Given $\epsilon>0$, $\zeta(1+\frac{\epsilon}{2}+it)=O(1)$. By the functional equation, $\zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} = O(|t|^{\frac{1}{2}+\frac{\epsilon}{2}})$ if $s=1+\frac{\epsilon}{2}+it$. Hence $\zeta(\frac{1}{2}+it)=O(|t|^{\frac{1}{4}+\epsilon})$. We call this convexity bound. If we assume Riemann hypothesis, we would have $\zeta(\frac{1}{2}+it) = O(|t|^\epsilon)$ for every $\epsilon>0$.

Assume that $L(s,\pi,r_i)$ is entire, namely $M(s,\pi)$ is holomorphic for $Re(s)>0$. In order to apply Phragmen-Lindelöf theorem to $L(s,\pi,r_i)$, we take a vertical strip, $a\le Re(s)\le b$ such that $L(s,\pi,r_i)$ is absolutely convergent for $Re(s)=b$, and hence $L(s,\pi,r_i)$ is bounded on $Re(s)=b$, and also by the functional equation, on $Re(s)=a$. If we can prove that $L(s,\pi,r_i)$ is of finite order, we can apply Phragmen-Lindelöf theorem. The starting point is the following proposition.

Proposition 13.5 (Harish-Chandra). *$(M(s,\pi)f,f')$ is a function of finite order for $Re(s)>0$.*

Proof. This is Lemma 38 of [HC]. □

Consider

$$M(s,\pi,w_0)f = \prod_{i=1}^m \frac{L(is,\pi,r_i)}{L(1+is,\pi,r_i)\epsilon(is,\pi,r_i)} \otimes N(s,\pi_v,w_0)f_v.$$

Proposition 13.6. *(1) If $v|\infty$, $(N(s,\pi_v,w_0)f_v, f'_v)$ is a rational function of s.*
(2) If $v<\infty$, $(N(s,\pi_v,w_0)f_v, f'_v)$ is a rational function of q_v^{-s}.
(3) $N(s,\pi_v,w_0)=id$ for almost all v.

We need the following theorem from complex analysis. (See [Ge-Sh, Theorem 5.7])

Theorem 13.7. *Let $f(z), h(z)$ be analytic functions of finite order in the half-plane $x\geq 0$, $z=x+iy$. Let $g(z)=\frac{f(z)}{h(z)}$.*

(1) *Assume $g(z)$ is analytic for $x\geq 0$ (for example, $h(z)$ has no zeros for $x\geq 0$). Then g is of finite order for $x\geq 0$.*
(2) *Suppose further that $g(z)$ is entire and there exists $\alpha\in\mathbb{R}$, $C>0$ such that $|g(-x+iy)|=Ce^{\alpha x}|g(x+iy)|$ for any $x>0$. Then g is of finite order.*

Hence $\frac{\otimes_v N(s,\pi_v,w_0)f_v}{\epsilon(is,\pi,r_i)}$ is a function of finite order and non-vanishing for $Re(s)\geq\frac{1}{2}$. Therefore, by Theorem 13.7 (1), $\prod_{i=1}^m \frac{L(is,\pi,r_i)}{L(1+is,\pi,r_i)}$ is a function of finite order for $Re(s)\geq\frac{1}{2}$. Now starting at $Re(s)=N_0>>0$, where $\prod_{i=1}^m L(is,\pi,r_i)$ is absolutely convergent so that it is bounded, and moving to the left, we obtain

Proposition 13.8. *$\prod_{i=1}^m L(is,\pi,r_i)$ is a function of finite order for $Re(s)\geq\frac{1}{2}$.*

Use induction on m. If $m=1$, it is clear from the above proposition and by the functional equation, $L(s,\pi,r_1)$ is of finite order for all of $\mathbb{C}$. Now if we have the following conjecture, we conclude that $L(s,\pi,r_1)$ is of finite order for $Re(s)\geq\frac{1}{2}$, and by the functional equation, we are done.

Conjecture. *$L(s,\pi,r_i)^{-1}$ is of finite order for $Re(s)\geq 1$.*

Since we cannot prove the conjecture, we need to use Theorem 13.7 (2). Induction hypothesis is: For all $i\geq 2$, $L(s,\pi,r_i)$ is of finite order for $Re(s)\geq\frac{1}{2}$. Then $L(s,\pi,r_1)$ is a ratio of two analytic functions of finite order for $Re(s)\geq\frac{1}{2}$. Now we apply Theorem 13.8 (2) with $g(z)=L(z+\frac{1}{2},\pi,r_1)$. By the functional equation, $L(s,\pi,r_1)=\epsilon(s,\pi,r_1)L(1-s,\pi,\tilde{r}_1)$. Here $\epsilon(s,\pi,r_1)=Ce^{\alpha s}$. Hence

$$L(\frac{1}{2}-z,\pi,\tilde{r}_1)=\overline{L(\frac{1}{2}-\bar{z},\pi,r_1)}=\overline{g(-\bar{z})}=\overline{g(-x+iy)}.$$

Therefore, $|g(-x+iy)|=Ce^{\alpha x}|g(x+iy)|$. So $L(s,\pi,r_1)$ is of finite order for all of $\mathbb{C}$.

So we have obtained

Theorem 13.9 (Gelbart-Shahidi [Ge-Sh]). *Suppose $L(s,\pi,r_i)$ is entire. Then it is bounded in vertical strips.*