11. Local L-functions and functional equations.

In this section, we summarize the main results in [Sh1]. Recall the definition of local coefficients: Let $I(\nu,\sigma)$ be the induced representation. Then

$$\lambda_\psi(\nu,\sigma) = C_\psi(\nu,\sigma,w)\lambda_\psi(w\nu,w\sigma)A(\nu,\sigma,w).$$

We know that $\lambda_\psi(\nu,\sigma)$ is entire and non-vanishing. Hence

Proposition 11.1. *$C_\psi(\nu,\sigma,w)A(\nu,\sigma,w)$ has no zeros.*

Recall the cocycle relation for the intertwining operator: Let $w = w_n\cdots w_1$, and

$$A(\nu,\sigma,\theta,w) = A(w_{n-1}\cdots w_1\nu, w_{n-1}\cdots w_1\sigma,\theta_n,w_n)\cdots A(\nu,\sigma,\theta,w_1).$$

Each of the factors is a rank-one operator, where $\theta_{i+1} = w_i\theta_i$, $\Omega_i = \theta_i\cup\{\alpha_i\}$, $w_i(\theta_i)\subset\Omega_i$, $w_i(\alpha_i)<0$.

Proposition 11.2. $C_\psi(\nu,\sigma,\theta,w) = C_\psi(w_{n-1}\cdots w_1\nu, w_{n-1}\cdots w_1\sigma,\theta_n,w_n)\cdots C_\psi(\nu,\sigma,\theta,w_1)$.

Suppose $\mathbf{P}=\mathbf{MN}$ is a maximal parabolic subgroup and σ is a representation of $\mathbf{M}(F)$.

Proposition 11.3 (Shahidi). *Suppose F is archimedean or σ is spherical, and $\phi: W_F\times SL_2(\mathbb{C})\longrightarrow {}^LM$ is the parametrization of π. Then*

$$C_\psi(s,\sigma,w_0) = \prod_{i=1}^m \gamma(is, r_i\circ\phi,\psi).$$

Hence if we set $\gamma(s,\sigma,r_i,\psi) = \gamma(s,r_i\circ\phi,\psi)$, and $L(s,\sigma,r_i) = L(s,r_i\circ\phi)$ and $\epsilon(s,\sigma,r_i,\psi) = \epsilon(s,r_i\circ\phi,\psi)$, then

$$C_\psi(s,\sigma,w_0) = \prod_{i=1}^m \gamma(is,\sigma,r_i,\psi),\quad \gamma(s,\sigma,r_i,\psi) = \epsilon(s,\sigma,r_i,\psi)\frac{L(1-s,\sigma,\tilde r_i)}{L(s,\sigma,r_i)}.$$

The Artin γ-factor satisfies $\gamma(s,\sigma,r_i,\psi)\gamma(1-s,\sigma,\tilde r_i,\bar\psi)=1$.

Proof. We only remark on the last equality. By the definition of γ-factors, we only need to verify $\epsilon(s,\sigma,r_i,\psi)\epsilon(1-s,\sigma,\tilde r_i,\bar\psi)=1$. This is the identity (3.4.7) in [Ta]. □

Let $\mathbf{P}_1\subset\mathbf{P}$, and $\mathbf{Q}=\mathbf{M}\cap\mathbf{P}_1$. Let σ be a representation of $\mathbf{M}(F)$ such that $\sigma\hookrightarrow Ind_Q^M\rho\otimes exp(\langle\Lambda_0,H_Q(\,)\rangle)$. Then

$$I(s,\sigma)\hookrightarrow Ind_{P_1}^G\rho\otimes exp(\langle s\tilde\alpha+\Lambda_0,H_{P_1}(\,)\rangle).$$

Let $A(s\tilde\alpha+\Lambda_0,w_0)$ be the intertwining operator for the induced representation $Ind_{P_1}^G\rho\otimes exp(\langle s\tilde\alpha+\Lambda_0,H_{P_1}(\,)\rangle)$. Then

$$A(s,\sigma,w_0) = A(s\tilde\alpha+\Lambda_0,\rho,w_0)|_{I(s,\sigma)}.$$

So $C_\psi(s,\sigma,w_0) = C_\psi(s\tilde\alpha+\Lambda_0,\rho,w_0)$. Since $\sigma\hookrightarrow Ind_Q^M\rho\otimes exp(\langle\Lambda_0,H_Q(\,)\rangle)$ for ρ supercuspidal, we first calculate $C_\psi(s,\rho,w_0)$ for ρ supercuspidal.

Recall first the following two propositions.

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Proposition 11.4 (Shahidi). *Suppose F is a p-adic field and ρ is ψ-generic supercuspidal representation of $\mathbf{M}(F)$. There exists a number field k such that $k_v = F$ and a globally generic cuspidal representation π of $\mathbf{M}(\mathbb{A})$ such that $\pi_v = \rho$ and π_w is spherical for all $w \neq v$, $w < \infty$.*

Proposition 11.5 (Shahidi). *For each $i \geq 2$, there exists a split reductive group $\mathbf{G}_i$ and a maximal parabolic subgroup $\mathbf{P}_i = \mathbf{M}_i\mathbf{N}_i$ and a cuspidal representation π' of $\mathbf{M}_i(\mathbb{A})$ such that $L_S(s, \pi, r_i) = L_S(s, \pi', r'_1)$, where $r' = \oplus_{j=1}^{m'} r'_j$, $m' < m$. If π_v is spherical, we can take π'_v to be spherical.*

Theorem 11.6 (Shahidi). *Suppose F is a p-adic field and ρ is ψ-generic supercuspidal representation of $\mathbf{M}(F)$. Then there exists a unique m complex functions $\gamma(s, \rho, r_i, \psi)$ such that*

$$C_\psi(s, \rho, w_0) = \prod_{i=1}^{m} \gamma(is, \rho, r_i, \psi).$$

It satisfies $\gamma(s, \rho, r_i, \psi)\gamma(1 - s, \rho, \tilde{r}_i, \bar{\psi}) = 1$.

Remark. If $\phi : W_F \longrightarrow {}^L M$ is the parametrization of ρ, then it is expected that $\gamma(s, \rho, r_i, \psi) = \gamma(s, r_i \circ \phi, \psi)$. But it is not proved yet, except for a few cases.

Proof. First, we prove the existence. Use induction on m. If $m = 1$, it is clear. Suppose the proposition is true for $m' < m$. By Proposition 11.4, there exists a number field k such that $k_v = F$ and a globally generic cuspidal representation π of $\mathbf{M}(\mathbb{A})$ such that $\pi_v = \rho$ and π_w is spherical for all $w \neq v$, $w < \infty$. By Proposition 11.5, for $i \geq 2$, there exists a split reductive group $\mathbf{G}_i$ and a maximal parabolic subgroup $\mathbf{P}_i = \mathbf{M}_i\mathbf{N}_i$ and a cuspidal representation π' of $\mathbf{M}_i(\mathbb{A})$ such that $L_S(s, \pi, r_i) = L_S(s, \pi', r'_1)$, where $r' = \oplus_{j=1}^{m'} r'_j$, $m' < m$, and $S = \{v\}$. Now we define $\gamma(s, \rho, r_i, \psi) = \gamma(s, \pi'_v, r'_1, \psi)$. Then we define

$$\gamma(s, \rho, r_1, \psi) = C_\psi(s, \rho, w_0) \prod_{i=2}^{m} \gamma(is, \rho, r_i, \psi)^{-1}.$$

Next, we prove the uniqueness. Recall the crude functional equation

$$\prod_{i=1}^{m} L_S(is, \pi, r_i) = \prod_{v \in S} C_{\psi_v}(s\tilde{\alpha}, \pi_v, w_0) \prod_{i=1}^{m} L_S(1 - is, \pi, \tilde{r}_i).$$

By induction on i as in Theorem 7.4, we obtain the functional equation

$$L_S(s, \pi, r_i) = \prod_{w \in S} \gamma(s, \pi_w, r_i, \psi_v) L_S(1 - s, \pi, \tilde{r}_i).$$

Then

$$\gamma(s, \rho, r_i, \psi_v) = \frac{L_S(s, \pi, r_i)}{L_S(1 - s, \pi, \tilde{r}_i)} \prod_{w \in S, w \neq v} \gamma(s, \pi_w, r_i, \psi_w).$$

By Proposition 11.3, $\gamma(s,\pi_w,r_i,\psi_w)$, $w\in S, w\neq v$, are Artin factors attached to π_w, r_i, and hence uniquely determined. The partial L-function is uniquely determined. Hence $\gamma(s,\rho,r_i,\psi_v)$ is uniquely determined by ρ, r_i.

We apply the functional equation twice, and obtain

$$L_S(s,\pi,r_i)=\prod_{w\in S}\gamma(s,\pi_w,r_i,\psi_v)L_S(1-s,\pi,\tilde{r}_i)=\prod_{w\in S}\gamma(s,\pi_w,r_i,\psi_v)\gamma(1-s,\pi_w,\tilde{r}_i,\bar{\psi}_v)L_S(s,\pi,r_i).$$

Hence $\prod_{w\in S}\gamma(s,\pi_w,r_i,\psi_v)\gamma(1-s,\pi_w,\tilde{r}_i,\bar{\psi}_v)=1$. If $w\neq v$, by Proposition 11.3, $\gamma(s,\pi_w,r_i,\psi_v)\gamma(1-s,\pi_w,\tilde{r}_i,\bar{\psi}_v)=1$. Therefore, $\gamma(s,\rho,r_i,\psi)\gamma(1-s,\rho,\tilde{r}_i,\bar{\psi})=1$. $\square$

For a general σ, let $\sigma\hookrightarrow Ind_Q^M\rho\otimes exp(\langle\Lambda_0,H_Q(\,)\rangle)$ for ρ supercuspidal. Then $C_\psi(s,\sigma,w_0)=C_\psi(s\tilde{\alpha}+\Lambda_0,\rho,w_0)$ which is a product of rank-one C_ψ's attached to supercuspidal representations by Proposition 11.2. Each rank-one C_ψ's is a product of γ-factors by Theorem 11.6. Now we define $\gamma(is,\sigma,r_i,\psi)$ to be the product of the γ-factors with coefficients is. For example, let σ be the unique subrepresentation of $Ind\,|det|^{\frac{1}{2}}\rho\otimes|det|^{-\frac{1}{2}}\rho$, where ρ is a supercuspidal representation of $GL_2(F)$. Consider σ to be a representation of $GL_4(F)\subset Sp_8(F)$. Then $C_\psi(s,\sigma,w_0)=C_\psi(s-\frac{1}{2},\rho)C_\psi(2s,\rho\otimes\rho)C_\psi(s+\frac{1}{2},\rho)$, where $C_\psi(s,\rho)$ is the local coefficient for $GL_2(F)\subset Sp_4(F)$, and $C_\psi(s,\rho\otimes\rho)$ is for $GL_2\times GL_2\subset GL_4$. Here $C_\psi(s,\rho)=\gamma(s,\rho)\gamma(2s,\omega_\rho)$, where ω_ρ is the central character of ρ. Hence we define

$$\gamma(s,\sigma,r_1,\psi)=\gamma(s-\frac{1}{2},\rho)\gamma(s+\frac{1}{2},\rho),\quad \gamma(2s,\sigma,r_2,\psi)=\gamma(2s,\rho\otimes\rho)\gamma(2s-1,\omega_\rho)\gamma(2s+1,\omega_\rho).$$

We prove

Theorem 11.7 (properties of γ-functions). *Let σ be an irreducible admissible generic representation of $\mathbf{M}(F)$. Then there exists an m complex functions $\gamma(s,\sigma,r_i,\psi)$ such that*

(1) $C_\psi(s,\sigma,w_0)=\prod_{i=1}^m\gamma(is,\sigma,r_i,\psi)$

(2) $\gamma(s,\sigma,r_i,\psi)\gamma(1-s,\sigma,\tilde{r}_i,\bar{\psi})=1$, *for each $i=1,...,m$*

(3) *(functional equation) Let $\pi=\otimes_v\pi_v$ be a generic cuspidal representation. Let S be a finite set of places such that for $v\notin S$, π_v,ψ_v are unramified. Then*
$$L_S(s,\pi,r_i)=\prod_{v\in S}\gamma(s,\pi_v,r_i,\psi_v)L_S(1-s,\pi,\tilde{r}_i).$$

(4) *(Multiplicativity of γ-factors) Suppose $\sigma\subset Ind_{M_\theta N_\theta}^M\sigma_1\otimes 1$, where σ_1 is an irreducible admissible representation of M_θ. Let $\theta'=w(\theta)\subset\Delta$ and fix a reduced decomposition $w=w_n\cdots w_1$. For each j, there exists a unique root $\alpha_j\in\Delta$ such that $w_j(\alpha_j)<0$. Let $\Omega_j=\theta_j\cup\{\alpha_j\}$. The group M_{Ω_j} contains $M_{\theta_j}N_{\theta_j}$ as a maximal parabolic subgroup, and $w_{j-1}\cdots w_1\sigma_1$ is a representation of M_{θ_j}. The L-group ${}^LM_\theta$ acts on V_i. Given an irreducible constituent of this action, there exists a unique j, $1\leq j\leq n$ which under $w_{j-1}\cdots w_1$ is equivalent to an irreducible constituent of the action of ${}^LM_{\theta_j}$*

on the Lie algebra of $^LN_{\theta_j}$. We denote by $i(j)$ the index of this subspace of the Lie algebra of $^LN_{\theta_j}$. Finally, let S_i denote the set of all such j's where S_i is a proper subset of $1 \le j \le n$. Then

$$\gamma(s,\sigma,r_i,\psi) = \prod_{j\in S_i} \gamma(s, w_{j-1}\cdots w_1(\sigma_1), r_{i(j)}, \psi) \tag{11.1}$$

Proof. (1) and (3) follow from the definition of γ-factors and the induction on m. (2) follows from (4) and the supercuspidal case (Theorem 11.6). Now we prove the multiplicativity of γ-factors. If σ_1 is a supercuspidal representation, it is a cosenquence of the definition. Suppose σ_1 is arbitrary. Then $\sigma_1 \hookrightarrow Ind_Q^{M_\theta}\rho \otimes exp(\langle \Lambda_0, H_Q(\,)\rangle)$ for ρ supercuspidal. Then $\sigma \subset Ind_Q^{M_\theta}\,\rho\otimes 1$. So $\gamma(s,\sigma,r_i,\psi)$ is a product of γ-factors attached to ρ. A similar statement is true for the right hand side of (11.1). Hence by the uniqueness of γ-factors, we have (11.1). □

Example 11.8. Suppose σ is the Steinberg representation given as the unique subrepresentation of

$$Ind\,|det|^{\frac{p-1}{2}}\rho\otimes|det|^{\frac{p-1}{2}-1}\rho\otimes\cdots\otimes|det|^{-\frac{p-1}{2}}\rho,$$

where ρ is a supercuspidal representation of GL_k. Then

$$I(s,\sigma\otimes\tilde\rho)\subset Ind\,|det|^{\frac{s}{2}+\frac{p-1}{2}}\rho\otimes|det|^{\frac{s}{2}+\frac{p-1}{2}-1}\rho\otimes\cdots\otimes|det|^{\frac{s}{2}-\frac{p-1}{2}}\rho\otimes|det|^{-\frac{s}{2}}\tilde\rho.$$

By multiplicativity of γ-factors,

$$\gamma(s,\sigma\times\tilde\rho,\psi)=\prod_{i=0}^{p-1}\gamma(s+\frac{p-1}{2}-i,\rho\times\tilde\rho,\psi).$$

Lemma 11.9. *Let π be an admissible representation of $M(F)$, F, p-adic. Let $\lambda_\psi(s,\pi)$ be the Whittaker functional for $I(s,\pi)$. Then for every $f\in I(s,\pi)$, $\lambda_\psi(s,\pi)(f)$ is a polynomial in q^s and q^{-s}.*

Proof. It is proved in [Sh8, Lemma 2.2] for the case of $GL_n\times GL_m\subset GL_{n+m}$. But in light of the result in [Ca-Sh, Lemma 2.2] the proof is general. □

Lemma 11.10. *Let π, F be as above. Let $A(s,\pi,w_0)$ be the intertwining operator for $I(s,\pi)$. Then for every $f\in I(s,\pi)$ and $g\in G(F)$, $A(s,\pi,w_0)f(g)$ is a rational function of q^{-s}.*

Proof. It is proved in [Sh8, Lemma 2.3] for the case of $GL_n\times GL_m\subset GL_{n+m}$. But in light of the result in [Si, Lemma 1.4, 1.5], the proof is general. □

Theorem 11.11. *$\gamma(s,\pi,r_i,\psi)$ is a rational function of q^{-s}.*

Proof. By the definition of local coefficients, $C_\psi(s,\pi,w_0)$ is a rational function of q^{-s}. Hence by induction, $\gamma(s,\pi,r_i,\psi)$ is a rational function of q^{-s}. □

11.1 Definition of local L-functions.

11.1.1 σ is tempered, generic. We defined $\gamma(s,\sigma,r_i,\psi)$ as a rational function of q^{-s}. Let $P_{\sigma,i}(t)$ be the unique polynomial satisfying $P_{\sigma,i}(0)=1$ such that $P_{\sigma,i}(q^{-s})$ is the numerator of $\gamma(s,\sigma,r_i,\psi)$. Define

$$L(s,\sigma,r_i)=P_{\sigma,i}(q^{-s})^{-1},\quad L(s,\sigma,\tilde{r}_i)=P_{\tilde{\sigma},i}(q^{-s})^{-1}.$$

Since $\gamma(s,\sigma,r_i,\psi)\gamma(1-s,\sigma,\tilde{r}_i,\bar{\psi})=1$,

$$\gamma(s,\sigma,r_i,\psi)\frac{L(s,\sigma,r_i)}{L(1-s,\sigma,\tilde{r}_i)},$$

is a monomial in q^{-s}, which we denote by $\epsilon(s,\sigma,r_i,\psi)$. Hence

$$\gamma(s,\sigma,r_i,\psi)=\epsilon(s,\sigma,r_i,\psi)\frac{L(1-s,\sigma,\tilde{r}_i)}{L(s,\sigma,r_i)}.$$

Example 11.12. Consider Example 11.9. In that case, $L(s,\sigma\times\tilde{\rho})=L(s+\frac{p-1}{2},\rho\times\tilde{\rho})$. Notice a lot of cancellations in the γ-factors. Also $L(1-s,\tilde{\sigma}\times\rho)=L(1-s+\frac{p-1}{2},\rho\times\tilde{\rho})$.

Conjecture 11.13 (Shahidi). *Let σ be tempered, generic. Then $L(s,\sigma,r_i)$ is holomorphic for $Re(s)>0$.*

Theorem 11.14. *The above conjecture is true except for 4 cases; E_7-3, E_8-3, E_8-4 and $(xxviii)(D_7\subset E_8)$. (These 4 cases have the Levi subgroups of type D_n or E_6.)*

11.1.2 σ non-tempered, generic. By Langlands' classification, we can write

$$\sigma\hookrightarrow Ind_Q^M\rho\otimes exp(\langle\Lambda_0,H_Q(\,)\rangle),$$

where ρ is generic, tempered. Then

$$I(s,\sigma)\hookrightarrow Ind_{P_1}^G\rho\otimes exp(\langle s\tilde{\alpha}+\Lambda_0,H_{P_1}(\,)\rangle).$$

By multiplicativity of γ-factors, $\gamma(s,\sigma,r_i,\psi)$ is a product of rank-one γ-factors for ρ. Define $L(s,\sigma,r_i)$ to be the product of rank-one L-functions for each γ-factors. We then define $\epsilon(s,\sigma,r_i,\psi)$ to satisfy $\gamma(s,\sigma,r_i,\psi)=\epsilon(s,\sigma,r_i,\psi)\frac{L(1-s,\sigma,\tilde{r}_i)}{L(s,\sigma,r_i)}$.

Example 11.15. Suppose $\sigma=\mu\circ det$ is a representation of $GL_2(F)$, where μ is an unramified character. It is a subrepresentation of $Ind\,\mu|\,|^{-\frac{1}{2}}\otimes\mu|\,|^{\frac{1}{2}}$. Then

$$I(s,\sigma)=Ind\,\pi|det|^{\frac{s}{2}}\otimes|\,|^{-\frac{s}{2}}\hookrightarrow Ind\,\mu|\,|^{\frac{s}{2}-\frac{1}{2}}\otimes\mu|\,|^{\frac{s}{2}+\frac{1}{2}}\otimes|\,|^{-\frac{s}{2}}.$$

Hence $\gamma(s,\sigma,\psi)=\gamma(s+\frac{1}{2},\mu,\psi)\gamma(s-\frac{1}{2},\mu,\psi)$, and $L(s,\sigma)=L(s+\frac{1}{2},\mu)L(s-\frac{1}{2},\mu)$.

On the other hand, if π is the Steinberg representation, which is the subrepresentation of $Ind\,\mu|\,|^{\frac{1}{2}}\otimes\mu|\,|^{-\frac{1}{2}}$, then $\gamma(s,\pi,\psi)=\gamma(s,\sigma,\psi)$, but $L(s,\pi)=L(s+\frac{1}{2},\mu)$.

Theorem 11.16 (the functional equations of the completed L-functions). *Let $\pi = \otimes_v \pi_v$ be a ψ-generic cuspidal representation. Let*

$$L(s,\pi,r_i) = \prod_{\text{all } v} L(s,\pi_v,r_i), \quad \epsilon(s,\pi,r_i) = \prod_{\text{all } v} \epsilon(s,\pi_v,r_i,\psi_v).$$

Then $L(s,\pi,r_i) = \epsilon(s,\pi,r_i)L(1-s,\pi,\tilde{r}_i)$.

11.2 Properties of local L-functions; supercuspidal representations.

Suppose σ is a generic supercuspidal representation of $\mathbf{M}(F)$, F p-adic. Then

(1) Unless $\mathbf{P}$ is self-conjugate, and $w_0(\sigma) \simeq \sigma$, $C_\psi(s\tilde{\alpha},\sigma,w_0)$ never vanishes. Hence if $\mathbf{P}$ is not self-conjugate, or $w_0(\sigma) \not\simeq \sigma$, $L(s,\sigma,r_i) = 1$ for all i. Since $L(s,\sigma,r_i)$ comes from non self-conjugate cases (except for the representation r_3 of $E_8 - 1$ case), $L(s,\sigma,r_i) = 1$ for all $i \geq 3$. (The representation r_3 of $E_8 - 1$ case appears as the first L-function in $E_6 - 2$ case, which is not self-conjugate.)

(2) Each $L(s,\sigma,r_i)$, $i = 1,2$, is a product of the form $\prod_j (1-\alpha_j q^{-s})^{-1}$, where $\alpha_j \in \mathbb{C}$, and $|\alpha_j| = 1$.

(3) $\frac{1}{\prod_{i=1}^m L(is,\sigma,r_i)} A(s,\sigma,w_0)$ is entire and non-zero.

(4) The following are equivalent: (a) $A(s,\sigma,w_0)$ has a pole at $s=0$; (b) either $L(s,\sigma,r_1)$ or $L(s,\sigma,r_2)$ has a pole at $s=0$, only one of them has a pole; (c) $I(0,\sigma)$ is irreducible and $w_0(\sigma) \simeq \sigma$.

(5) Suppose $I(s_0,\sigma)$ is reducible for $s_0 > 0$. Then $s_0 = \frac{1}{2}$ or 1. And $J(s,\sigma)$ is unitary for $0 \leq s \leq s_0$ is unitary and non-unitary for $s > s_0$. The unique subrepresentation of $I(s_0,\sigma)$ is square integrable.

Example 11.17. Let $\mathbf{G} = GL_{2n}$ and $\mathbf{M} \simeq GL_n \times GL_n$. Let σ be an irreducible generic supercuspidal representation of $GL_n(F)$. Then $L(s,\sigma\times\tilde{\sigma}) = (1-q^{-rs})^{-1}$, where r is the order of the cyclic group $\{\eta : \sigma\otimes\eta \simeq \sigma\}$. Here $\eta^n = 1$ and $r|n$.