5. Eisenstein series and constant terms.

5.1 Definition of Eisenstein series. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, and $\mathbf{P} = P_\theta = \mathbf{MN}$, where $\theta = \Delta - \{\alpha\}$. For $f_s \in I(s,\pi)$, define the Eisenstein series

$$E(s,\pi,f_s,g) = \sum_{\gamma\in\mathbf{P}(F)\backslash\mathbf{G}(F)} f_s(\gamma g).$$

If $f = \phi exp(\langle s\tilde{\alpha} + \rho_P, H_P(\,)\rangle)$, then we denote it by

$$E(s,\pi,\phi,g) = \sum_{\gamma\in\mathbf{P}(F)\backslash\mathbf{G}(F)} \phi(\gamma g) exp(\langle s\tilde{\alpha} + \rho_P, H_P(\gamma g)\rangle).$$

Example 5.1. Let $\mathbf{G} = GL_2$, and $\mathbf{P} = \mathbf{B}$, the Borel subgroup. By the strong approximation, $GL_2(\mathbb{A}_\mathbb{Q}) = GL_2(\mathbb{Q})GL_2^+(\mathbb{R})K$, where $K = \prod_p K_p$, where $K_p = GL_2(\mathbb{Z}_p)$ for all p, and $K_\infty = SO(2)$. Then $GL_2(\mathbb{Q}) \cap GL_2^+(\mathbb{R})K = SL_2(\mathbb{Z})$. (If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \cap GL_2(\mathbb{Q})$, then $a,b,c,d \in \mathbb{Z}_p \cap \mathbb{Q}$ for all p. Hence $a,b,c,d \in \mathbb{Z}$, and $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$.) Hence $GL_2(\mathbb{Q}) = SL_2(\mathbb{Z}) \cdot \mathbf{B}(\mathbb{Q})$. (Note that $GL_2(\mathbb{Q}) \subset GL_2^+(\mathbb{R}) \cdot K \cdot \mathbf{B}(\mathbb{Q})$.) So

$$\mathbf{B}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q}) \simeq (SL_2(\mathbb{Z}) \cap \mathbf{B}(\mathbb{Q}))\backslash SL_2(\mathbb{Z}).$$

Now take $\phi = 1$ and $g = \begin{pmatrix} y^{\frac{1}{2}} & y^{-\frac{1}{2}}x \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$, $y > 0$. Then by identifying $\mathfrak{a}$ with $\mathbb{R}$, we see that $\tilde{\alpha} = \rho_P$, $H_P(g) = \log y$ and $exp(\langle s\tilde{\alpha} + \rho_P, H_P(g)\rangle) = y^{\frac{s+1}{2}}$. Note that $g \cdot i = z = x + yi$ and $Im(\gamma z) = \frac{Im z}{|cz+d|^2}$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\gamma z = \frac{az+b}{cz+d}$. Hence

$$E(s,1,g) = \sum_{\gamma\in(SL_2(\mathbb{Q})\cap\mathbf{B}(\mathbb{Q}))\backslash SL_2(\mathbb{Z})} Im(\gamma z)^{\frac{s+1}{2}} = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ gcd(c,d)=1}} \frac{y^{\frac{s+1}{2}}}{|cz+d|^{s+1}}.$$

If we use the fact that every non-zero $(m,n) \in \mathbb{Z}^2$ can be written as $\alpha \cdot (c,d)$, where $\alpha \in \mathbb{Z}_+$, $gcd(c,d) = 1$, then

$$E(s,1,g) = \frac{1}{\zeta(s+1)} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^{s+1}} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ gcd(c,d)=1}} \frac{y^{\frac{s+1}{2}}}{|cz+d|^{s+1}} = \frac{1}{\zeta(s+1)} \sum_{(m,n)\neq(0,0)} \frac{y^{\frac{s+1}{2}}}{|mz+n|^{s+1}}.$$

This is the classical Eisenstein series.

Theorem 5.2 (Langlands; Basic properties of Eisenstein series).

(1) $E(s,\pi,f_s,g)$ *is absolutely convergent for* $Re(s) > \langle\rho_P,\alpha\rangle = \frac{2(\rho_P,\alpha)}{(\alpha,\alpha)}$.

(2) $E(s,\pi,f_s,g)$ *and* $M(s,\pi)$ *have meromorphic continuation to all of* $\mathbb{C}$ *and satisfies a functional equation*

$$E(-s,w_0(\pi),M(s,\pi)f_s,g) = E(s,\pi,f_s,g); \quad M(-s,w_0(\pi))M(s,\pi) = id.$$

(3) $E(s,\pi,f_s,g)$ *and* $M(s,\pi)$ *are holomorphic on* $Re(s) = 0$.

(4) *(adjoint formula) Suppose* $\mathbf{P}$ *is self-conjugate, and consider* $M(s,\pi)$ *as an intertwining operator for* $\mathcal{H}$. *Then*

$$(M(s,\pi)\phi_1,\phi_2) = (\phi_1, M(\bar{s},\pi)\phi_2).$$

Namely, $M(s,\pi)^* = M(\bar{s},\pi)$.

(5) *The singularities of* $E(s,\pi,\phi,g)$ *and* $M(s,\pi)$ *are the same. In the region* $Re(s) > 0$, *there are only finitely many of them, all are simple and on the interval,* $(0,\langle\rho_P,\alpha\rangle)$. *(Here note that we normalized* π *so that it is trivial on* $\mathbf{A}(\mathbb{R})_+$.*)*

(6) $(M(s,\pi)\phi_1,\phi_2)$ *is bounded in vertical strips* $T_{\epsilon,I} = \{z\in\mathbb{C} |\, Re(z)\in I, Im(z)\geq \epsilon\}$, *where* $\epsilon > 0$ *and* I *is a closed positive real interval.*

5.2 Constant terms. Define the constant term of $E(s,\pi,\phi,g)$ along a parabolic subgroup $\mathbf{Q} = \mathbf{M}_Q\mathbf{N}_Q$ by

$$E_Q(s,\pi,\phi,g) = \int_{\mathbf{N}_Q(F)\backslash\mathbf{N}_Q(\mathbb{A})} E(s,\pi,\phi,ng)\,dn.$$

Theorem 5.3. *Unless* $\mathbf{Q} = \mathbf{P},\mathbf{P}'$, $E_Q(s,\pi,\phi,g) = 0$. *If* $\mathbf{P}$ *is self-conjugate, i.e.,* $\mathbf{P} = \mathbf{P}'$, *then*

$$E_P(s,\pi,\phi,g) = \phi(g)exp(\langle s\tilde{\alpha}+\rho_P, H_P(g)\rangle) + M(s,\pi)\phi(g)exp(\langle -s\tilde{\alpha}+\rho_P, H_P(g)\rangle).$$

If $\mathbf{P}$ *is not self-conjugate, then*

$$E_P(s,\pi,\phi,g) = \phi(g)exp(\langle s\tilde{\alpha}+\rho_P, H_P(g)\rangle),$$
$$E_{P'}(s,\pi,\phi,g) = M(s,\pi)\phi(g)exp(\langle -s\tilde{\alpha}+\rho_P, H_P(g)\rangle)$$

Proof. Use Bruhat decomposition: Let $\theta_1,\theta_2\subset\Delta$ and let $\mathbf{P}_i = P_{\theta_i}$ for $i=1,2$. Then

$$\mathbf{G}(F) = \bigcup_{w\in W_{\theta_1}\backslash W/W_{\theta_2}} \mathbf{P}_1(F)w^{-1}\mathbf{P}_2(F),$$

where W_{θ_i} is the subgroup of W generated by $\{w_\alpha |\, \alpha\in\theta_i\}$. We use the following three facts from Casselman:

(1) There are canonical double coset representatives in $W_{\theta_1}\backslash W/W_{\theta_2}$. They are given by $W(\theta_1,\theta_2) = \{w\in W |\, w(\alpha) > 0$ for $\alpha\in\theta_1$ and $w^{-1}(\beta) > 0$ for $\beta\in\theta_2\}$.

(2) The product map induces an isomorphism

$$P_{\theta_1} \times \{w^{-1}\} \times \prod_{\substack{\alpha\in\Phi_+-\Sigma^+_{\theta_2}\\ w^{-1}(\alpha)\notin\Phi_+-\Sigma^+_{\theta_1}}} U_\alpha \simeq P_{\theta_1} w^{-1} P_{\theta_2}.$$

(3) The canonical projection induces an isomorphism

$$\prod_{\substack{\alpha\in\Phi_+-\Sigma^+_{\theta_2}\\ w^{-1}(\alpha)\notin\Phi_+-\Sigma^+_{\theta_1}}} U_\alpha \simeq (wN_{\theta_1}w^{-1}\cap N_{\theta_2})\backslash N_{\theta_2}.$$

Apply this to $\theta=\theta_1$. Let $\mathbf{N}_1 = w\mathbf{N}w^{-1}\cap N_{\theta_2}$.

$$E_{P_{\theta_2}}(s,\pi,\phi,g) = \int_{N_{\theta_2}(F)\backslash N_{\theta_2}(\mathbb{A})} \sum_{\gamma\in\mathbf{P}(F)\backslash\mathbf{G}(F)} \phi(\gamma ng)exp(\langle s\tilde\alpha+\rho_P, H_P(\gamma ng)\rangle)\, dn$$

$$= \sum_{w\in W(\theta,\theta_2)} \sum_{n'\in\mathbf{N}_1(F)\backslash N_{\theta_2}(F)} \int_{N_{\theta_2}(F)\backslash N_{\theta_2}(\mathbb{A})} \phi(w^{-1}n'ng)exp(\langle s\tilde\alpha+\rho_P, H_P(w^{-1}n'ng)\rangle)\, dn$$

$$= \sum_{w\in W(\theta,\theta_2)} \int_{\mathbf{N}_1(F)\backslash N_{\theta_2}(\mathbb{A})} \phi(w^{-1}ng)exp(\langle s\tilde\alpha+\rho_P, H_P(w^{-1}ng)\rangle)\, dn$$

Here $m\longmapsto \phi(mg)$ belongs to the space of π. So it is a cusp form. Note that $w^{-1}P_{\theta_2}w\cap\mathbf{M}$ is parabolic in $\mathbf{M}$ with unipotent radical $w^{-1}N_{\theta_2}w\cap\mathbf{M}$. Hence if $w^{-1}N_{\theta_2}w\cap\mathbf{M}\neq 1$,

$$\int_{\mathbf{N}_1(F)\backslash N_{\theta_2}(\mathbb{A})} \phi(w^{-1}ng)exp(\langle s\tilde\alpha+\rho_P, H_P(w^{-1}ng)\rangle)\, dn = 0,$$

since $\int_{\mathbf{N}''(F)\backslash\mathbf{N}''(\mathbb{A})} \phi(w^{-1}ng)exp(\langle s\tilde\alpha+\rho_P, H_P(w^{-1}ng)\rangle)\, dn=0$, where $\mathbf{N}''=w\mathbf{M}w^{-1}\cap N_{\theta_2}$.

If $w^{-1}N_{\theta_2}w\cap\mathbf{M}=1$, $w\mathbf{M}w^{-1}\cap N_{\theta_2}=1$. Since $w(\theta)>0$, $w(\theta)\subset\Sigma^+_{\theta_2}$. So $w(\Sigma^+_\theta)\subset\Sigma^+_{\theta_2}$. Hence $w\mathbf{M}w^{-1}\subset M_{\theta_2}$. Since $\mathbf{M}$ is a maximal Levi subgroup, $w\mathbf{M}w^{-1}=M_{\theta_2}$. This implies that $w(\theta)=\theta_2$. (If $w(\alpha)=\beta+\gamma$, where $\beta,\gamma\in\Sigma^+_{\theta_2}$, then $\alpha=w^{-1}(\beta)+w^{-1}(\gamma)$ with $w^{-1}(\beta),w^{-1}(\gamma)>0$. This contradicts to the fact that α is simple.) If $\mathbf{P}$ is self-conjugate, there are two elements, namely $w=1,w_0$ such that $w(\theta)=\theta$. Here we normalize the measure on the compact set $\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})$ so that $\int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} dn=1$. So if $w=1$, it gives rise to $\phi(g)exp(\langle s\tilde\alpha+\rho_P, H_P(g)\rangle)$. If $w=w_0$, we obtain

$$\int_{\mathbf{N}(\mathbb{A})} \phi(w_0^{-1}ng)exp(\langle s\tilde\alpha+\rho_P, H_P(w_0^{-1}ng)\rangle)\, dn$$

which is exactly $M(s,\pi)\phi(g)exp(\langle -s\tilde{\alpha}+\rho_P, H_P(w_0^{-1}ng)\rangle)$. The other case is similar.

Let us illustrate the above proof with an example: Let $\mathbf{G}=Sp(4), \mathbf{P}=P_\theta$, $\theta=\{e_1-e_2\}$. Then $\mathbf{M}\simeq GL_2$, and $W(\theta,\theta)=\{1,c_2,w_0=(1\,2)c_1c_2\}$, where c_1,c_2 are sign changes. If $w=c_2$, then $w^{-1}\mathbf{N}w\cap\mathbf{M}=U_{e_1-e_2}=\{diag(\begin{pmatrix}1&x\\0&1\end{pmatrix},\begin{pmatrix}1&-x\\0&1\end{pmatrix})\}$.

5.3 Pseudo-Eisenstein series. Note that $\mathbf{N}(\mathbb{A})\mathbf{M}(F)\backslash\mathbf{G}(\mathbb{A})=(\mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A})^1)\cdot\mathbf{A}(\mathbb{R})_+\cdot K$. Then $exp(\langle s\tilde{\alpha},H_P(y)\rangle)=y^s$ for $y\in\mathbf{A}(\mathbb{R})_+\simeq\mathbb{R}_+$. We choose measures dm_1 on $\mathbf{M}(\mathbb{A})^1$ and da on $\mathbf{A}(\mathbb{R})_+$ (multiplicative measure, i.e., $\frac{dy}{y}$, where dy is a measure on $\mathbb{R}$) so that

$$\int_{\mathbf{G}(\mathbb{A})} f(g)\,dg=\int_{\mathbf{N}(\mathbb{A})}\int_{\mathbf{M}(\mathbb{A})^1}\int_{\mathbf{A}(\mathbb{R})_+}\int_K f(nm_1ak)e^{-\langle 2\rho_P,H_P(a)\rangle}\,dkdadm_1dn.$$

Now $E(s,\pi,\phi,g)$ is an automorphic form (see the lecture by J. Cogdell for the precise definition) for any s, and for the trace formula, we need $E(s,\pi,\phi,g)$ for $Re(s)=0$. Since non-constant terms are rapidly decreasing, they are square integrable. However, since the integral $\int_1^\infty y^s\,dy$ is convergent only for $s<-1$, $E(s,\pi,\phi,g)$ is not square integrable at $Re(s)=0$. It is integrable if $-\langle\rho_P,\alpha\rangle<Re(s)<\langle\rho_P,\alpha\rangle$. In order to obtain square integrable automorphic forms, we need to multiply ϕ by special type of functions, namely, Paley-Wiener type functions (Fourier transform of compactly supported functions on $\mathbb{R}_+$).

Consider $\Phi(g,s)=\phi(g)h(s)$, where $\phi\in\mathcal{H}$ and h is a Schwartz function, namely,

$$h(s)=\int_0^\infty \hat{h}(y)y^{-s}\frac{dy}{y},$$

where $\hat{h}\in C_c^\infty(\mathbb{R}_+)$. Now let $y=e^x$ and $s=\sigma+it$. Then

$$h(s)=\int_{-infty}^\infty \hat{h}(e^x)e^{-x(\sigma+it)}\,dx.$$

By Fourier inversion formula,

$$\hat{h}(e^x)e^{-x\sigma}=\frac{1}{2\pi}\int_{-\infty}^\infty h(s)e^{ixt}\,dt.$$

So we have

$$\hat{h}(y)=\frac{1}{2\pi}\int_{-\infty}^\infty h(s)y^s\,dt=\frac{1}{2\pi i}\int_{Re(s)=\sigma>>0} h(s)y^s\,ds,$$

where $s=\sigma+it$ (note that $ds=idt$).

Let $\hat{\Phi}(g) = \phi(g)\hat{h}(exp(\langle\tilde{\alpha}_P, H_P(g)\rangle))$. Here note that $exp(\langle\tilde{\alpha}, H_P(g)\rangle) = y$, where $y \in \mathbf{A}(\mathbb{R})_+$, in the decomposition $\mathbf{N}(\mathbb{A})\mathbf{M}(F)\backslash\mathbf{G}(\mathbb{A}) = \mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A})^1 \cdot \mathbb{A}(\mathbb{R})_+ \cdot K$. We define the pseudo-Eisenstein series

$$\theta_{\hat{\Phi}}(g) = \sum_{\gamma\in\mathbf{P}(F)\backslash\mathbf{G}(F)} \hat{\Phi}(\gamma g)exp(\langle\rho_P, H_P(\gamma g)\rangle) = \frac{1}{2\pi i}\int_{Re(s)=\sigma>>0} h(s)E(s,\pi,\phi,g)\,ds.$$

Here $\theta_{\hat{\Phi}}$ converges absolutely and $\theta_\Phi \in L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)$.

Sometimes, we put together ϕ and h: $\Phi(g,s)$ is the Fourier transform:

$$\Phi(g,s) = \int_0^\infty \hat{\Phi}(yg)exp(\langle -s\tilde{\alpha} - \rho_P, H_P(y)\rangle)\,dy,$$

where $y \in \mathbf{A}(\mathbb{R})_+$ and $\hat{\Phi}$ is a function on $\mathbf{N}(\mathbb{A})\mathbf{M}(F)\backslash\mathbf{G}(\mathbb{A})$ such that $y \longmapsto \hat{\Phi}(yg)$ is compactly supported in $\mathbf{A}(\mathbb{R})_+$. For each s, $\Phi(g,s) \in I(s,\pi)$. Now let $y = e^x$ and $s = \sigma + it$. Then

$$\Phi(g,s) = \int_{-\infty}^\infty \hat{\Phi}(e^x g)exp(\langle -\rho_P, H_P(e^x)\rangle)e^{-x(\sigma+it)}\,dx.$$

By Fourier inversion formula,

$$\hat{\Phi}(e^x g)exp(\langle -\rho_P, H_P(e^x)\rangle)e^{-x\sigma} = \frac{1}{2\pi}\int_{-\infty}^\infty \Phi(g,s)e^{ixt}\,dt.$$

By letting $x = 0$, we have

$$\hat{\Phi}(g) = \frac{1}{2\pi}\int_{-\infty}^\infty \Phi(g,s)\,dt = \frac{1}{2\pi i}\int_{Re(s)=\sigma>>0} \Phi(g,s)\,ds,$$

where $s = \sigma + it$. Let

$$\theta_{\hat{\Phi}}(g) = \sum_{\gamma\in\mathbf{P}(F)\backslash\mathbf{G}(F)} \hat{\Phi}(\gamma g) = \frac{1}{2\pi i}\int_{Re(s)=\sigma>>0} E(s,\pi,\Phi,g)\,ds.$$

We can define $\theta_{\hat{\Phi}}$ for all parabolic subgroups. Let $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)_{(M,\pi)}$ be the closure of the subspace of $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)$ generated by all such $\theta_{\hat{\Phi}}$. We put an equivalence relation on (M,π), namely, $(M_1,\pi_1) \sim (M_2,\pi_2)$ if there exists $w \in W$ such that $w(M_1) = M_2$ and $w(\pi_1) \simeq \pi_2$. Then

Theorem 5.4 (Langlands).

(1) $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega) = \oplus_{(M,\pi)} L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)_{(M,\pi)}$.

(2) $L^2_{dis}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)_{(M,\pi)}$, *the discrete spectrum, is spanned by residues of Eisenstein series. If* $\mathbf{P}$ *is maximal, they are the residues of* $E(s,\pi,\Phi,g)$ *for* $0 < s \leq \langle\rho_P,\alpha\rangle$. *(We move the contour* $Re(s) = \sigma >> 0$ *to* $Re(s) = 0$.*)*

(3) $L^2_{cont}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)_{(M,\pi)}$, *the continous spectrum. If* $\mathbf{P}$ *is maximal, it is spanned by*

$$\frac{1}{2\pi i}\int_{Re(s)=0} E(s,\pi,\Phi,g)\,ds.$$

(4) *Let* $L^2_{dis}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)$ *be the discrete spectrum. Then*

$$\begin{aligned} L^2_{dis}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega) &= \oplus_{(M,\pi)} L^2_{dis}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)_{(M,\pi)} \\ &= L^2_{cusp}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega) \oplus L^2_{res}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega), \end{aligned}$$

where $L^2_{res}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega) = \oplus_{(M,\pi),M\neq G} L^2_{dis}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)_{(M,\pi)}$.

(5) *(inner product of two pseudo-Eisenstein series) Suppose* $\mathbf{P}$ *is maximal.*

$$\begin{aligned} \langle \theta_{\hat\Phi_1}, \theta_{\hat\Phi_2}\rangle &= \int_{\mathbf{Z}(\mathbb{A})\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})} \theta_{\hat\Phi_1}(g)\overline{\theta_{\hat\Phi_2}(g)}\,dg \\ &= \frac{1}{2\pi i}\int_{Re(s)=\sigma>>0} \sum_{w\in W(\pi_1,\pi_2)} (M(s,\pi,w)\Phi_1(s), \Phi_2(-w\bar s))\,ds, \end{aligned}$$

where $W(\pi_1,\pi_2)$ *is the set of Weyl group elements such that* $w(M_1,\pi_1) = (M_2,\pi_2)$. *Hence* $W(\pi_1,\pi_2) = 1$ *or* $\{1, w_0\}$. *Especially,* L^2*-norm of* $\theta_{\hat\Phi}$ *is*

$$\begin{aligned} ||\theta_{\hat\Phi}||^2 &= \int_{\mathbf{Z}(\mathbb{A})\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})} |\theta_{\hat\Phi}(g)|^2\,dg \\ &= \frac{1}{2\pi i}\int_{Re(s)=\sigma>>0} \sum_{w\in W(\pi,\pi)} (M(s,\pi,w)\Phi(s), \Phi(-w\bar s))\,ds. \end{aligned}$$

Corollary 5.5. *Unless* $\mathbf{P}$ *is self-conjugate and* $w_0\pi \simeq \pi$,

$$L^2_{dis}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)_{(M,\pi)} = 0.$$

i.e., the Eisenstein series has no poles for $Re(s)\geq 0$, *and* $M(s,\pi)$ *is holomorphic for* $Re(s)>0$.

Proof. If $\mathbf{P}$ is not self-conjugate, $W(\pi,\pi)=1$. If $\mathbf{P}$ is self-conjugate and $w_0\pi \not\simeq \pi$, then $M(s,\pi,w_0)\Phi(s)\in I(-s,w_0\pi)$, but $\Phi(-w_0\bar s)\in I(\bar s,\pi)$. Hence again $W(\pi,\pi) = 1$. So the integrand in $||\theta_{\hat\Phi}||^2$ is just $(\Phi(s),\Phi(-\bar s))$, which is entire. Hence we can deform the contour $Re(s)=\sigma$ to $Re(s)=0$ without picking up residues. Hence the Eisenstein series has no poles for $Re(s)\geq 0$. Since the poles of the Eisenstein series coincide with those of $M(s,\pi)$, $M(s,\pi)$ is holomorphic for $Re(s)>0$. □

If s_0 is a pole of $E(s,\pi,\Phi,g)$, let $E_{-1}(\pi,\Phi,g) = res_{s=s_0}E(s,\pi,\Phi,g)$, and $M_{-1}\Phi = res_{s=s_0}M(s,\pi,w_0)$. Then $E_{-1}(\pi,\Phi,g)$ is a square integrable automorphic form and

$$\langle E_{-1}(\pi,\Phi,g), E_{-1}(\pi,\Phi,g)\rangle_{\mathbf{Z}(\mathbb{A})\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})} = (M_{-1}\Phi,\Phi)_{(\mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A})^1)\cdot K}.$$

Remark 5.6. Eisenstein series can be defined for residual representations. We call them residual Eisenstein series. For example, let $\mathbf{G} = Sp(2n)$, $\mathbf{P} = \mathbf{MN}$, where $\mathbf{M} \simeq GL_n$. Let χ be a grössencharacter of F. We can consider χ as a character of $\mathbf{M}(\mathbb{A})$ by setting $\chi(g) = \chi(detg)$. Then χ is a residual representation, namely, $\chi \in L^2_{dis}(\mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A}))_{(B_M,\chi)}$, where B_M is a Borel subgroup of $\mathbf{M}$. This follows from the fact that χ is the Langlands' quotient of $Ind\,\chi|\,|^{\frac{n-1}{2}} \otimes \chi|\,|^{\frac{n-1}{2}-1} \otimes \cdots \otimes \chi|\,|^{-\frac{n-1}{2}}$. We can form an Eisenstein series $E(s,\chi,\Phi)$. It is a generalization of Siegel Eisenstein series

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |det(CZ+D)|^{-s},$$

where $\Gamma = Sp(2n,\mathbb{Z})$, $\Gamma_\infty = \Gamma \cap \mathbf{P}(\mathbb{Q})$, and $Z = X + iY$, Y is a positive definite symmetric matrix.

We can show that $E(s,\chi,\Phi)$ is an iterated residue of Eisenstein series from Borel subgroup.