6. L-functions in the constant terms.

Recall the global intertwining operator $M(s,\pi) = \otimes_v A(s,\pi_v,w_0)$. For $f \in V(s,\pi)$,

$$f \in \otimes_{v\in S} V(s,\pi_v) \otimes \otimes_{v\notin S} f_v^0,$$

where S is a finite set of places, including archimedean places, and f_v^0 is the spherical vector such that $f_v^0(k_v) = 1$ for $k_v \in \mathbf{G}(\mathcal{O}_v)$. Suppose $f = \otimes f_v$, where $f_v = f_v^0$ for $v \notin S$. Then

$$M(s,\pi)f = \otimes_{v\in S} A(s,\pi_v,w_0)f_v \otimes \otimes_{v\notin S} A(s,\pi_v,w_0)f_v^0.$$

Langlands observed that L-functions show up in the calculation of $A(s,\pi_v,w_0)f_v^0$. First we need inducing in stages of induced representations. Suppose $\mathbf{P}_1 = \mathbf{M}_1\mathbf{N}_1 \subset \mathbf{P}_2 = \mathbf{M}_2\mathbf{N}_2$ are two parabolic subgroups. Then $\mathbf{P}_1 \longmapsto \mathbf{M}_2 \cap \mathbf{P}_1$ is a bijection from $\{\mathbf{P}_1 : \mathbf{P}_1 \subset \mathbf{P}_2\}$ onto the set of standard parabolic subgroups of $\mathbf{M}_2$. Then $\mathbf{M}_2 \cap \mathbf{N}_1$ is the unipotent radical of $\mathbf{M}_2 \cap \mathbf{P}_1$; $\mathbf{M}_2 \cap \mathbf{P}_1 = \mathbf{M}_1 \cdot (\mathbf{M}_2 \cap \mathbf{N}_1)$ is the Levi decomposition.

Lemma 6.1 (inducing in stages). *Suppose $\mathbf{P}_1 \subset \mathbf{P}_2$. Let $\mathbf{Q} = \mathbf{M}_2 \cap \mathbf{P}_1$. Let π be a representation of $\mathbf{M}_2(F)$ such that $\pi = Ind_Q^{M_2}\sigma \otimes exp(\langle \Lambda_0, H_Q(\,)\rangle)$. Then*

$$Ind_{P_2}^G \pi \otimes exp(\langle \Lambda, H_{P_2}(\,)\rangle) = Ind_{P_1}^G \sigma \otimes exp(\langle \Lambda + \Lambda_0, H_{P_1}(\,)\rangle),$$

where $\Lambda \in X^*(M_2) \otimes \mathbb{C}$ extends to $X^*(M_1) \otimes \mathbb{C}$.

Suppose π_v is spherical. Then $\pi_v \hookrightarrow I(\chi_v)$. By inducing in stages,

$$I(s,\pi_v) \subset Ind_B^G \chi_v \otimes exp < s\tilde{\alpha}, H_B(\,) >= I(s\tilde{\alpha},\chi_v).$$

We denote by $A(s\tilde{\alpha},\chi_v,w_0)$ the intertwining operator for $I(s\tilde{\alpha},\chi_v)$. Then $A(s,\pi_v,w_0) = A(s\tilde{\alpha},\chi_v,w_0)|_{I(s,\pi_v)}$. Here $f_v^0 \in I(s\tilde{\alpha},\chi_v)$, and

$$f_v^0(tuk) = \chi(t) exp < s\tilde{\alpha} + \rho_B, H_B(t) >,$$

for $t \in \mathbf{T}(F_v)$, $u \in \mathbf{U}(F_v)$, $k \in \mathbf{G}(\mathcal{O}_v)$. We need to calculate

$$A(s\tilde{\alpha},\chi_v,w_0)f_v^0(e) = \int_{\mathbf{N}(F_v)} f_v^0(w_0^{-1}n)\, dn.$$

(for simplicity, we assume that $\mathbf{P}$ is self-conjugate.)

We reduce the calculation to SL_2 case by using cocycle relation of intertwining operators: Let $\Lambda \in \mathfrak{a}_{\mathbb{C}}^*$, χ a character of $\mathbf{T}(F)$. Let $A(\Lambda,\chi,w)$ be the intertwining operator from $I(\Lambda,\chi)$ to $I(w\Lambda, w\chi)$. Let $w = w_2w_1$ be a minimal length decomposition, i.e., $l(w) = l(w_1) + l(w_2)$. Then

Theorem 6.2. $A(\Lambda,\chi,w) = A(w_1\Lambda, w_1\chi, w_2)A(\Lambda,\chi,w)$.

Hence if $w = w_{\alpha_n}\cdots w_{\alpha_1}$, where $\alpha_1, ..., \alpha_n \in \Delta$ and $l(w) = n$, then

$$A(\Lambda,\chi,w) = A(w_{\alpha_{n-1}}\cdots w_{\alpha_1}\Lambda, w_{\alpha_{n-1}}\cdots w_{\alpha_1}\chi, w_{\alpha_n})\cdots A(\Lambda,\chi,w_{\alpha_1}).$$

Each $A(w_{\alpha_i}\cdots w_{\alpha_1}\Lambda, w_{\alpha_i}\cdots w_{\alpha_1}\chi, w_{\alpha_{i+1}})$ is an intertwining operator for SL_2: Suppose α is a simple root. Recall the homomorphism $\phi_\alpha : SL_2 \longrightarrow \mathbf{G}$ such that

$$\phi_\alpha\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = e_\alpha(u), \quad \phi_\alpha\begin{pmatrix}u & 0\\ 0 & u^{-1}\end{pmatrix} = h_\alpha(u), \quad \phi_\alpha\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = w_\alpha.$$

Note that $exp(\langle\Lambda, H_B(h_\alpha(u))\rangle) = |u|^{\langle\Lambda,\alpha^\vee\rangle}$. Also $\langle w\Lambda,\alpha^\vee\rangle = \langle\Lambda,(w^{-1}\alpha)^\vee\rangle$ and $exp(\langle w\Lambda, H_B(h_\alpha(u))\rangle) = exp(\langle\Lambda, H_B(h_{w^{-1}\alpha}(u))\rangle)$.

If $w = w_2w_1$, $l(w) = l(w_2) + l(w_1)$, then

$$\{\alpha > 0, w\alpha < 0\} = w_1^{-1}\{\beta > 0, w_2\beta < 0\} \cup \{\gamma > 0, w_1\gamma < 0\},$$

where it is a disjoint union. Hence by induction, if $w = w_{\alpha_n}\cdots w_{\alpha_1}$, where $\alpha_1, ..., \alpha_n \in \Delta$, $l(w) = n$, then

$$\{\alpha > 0, w\alpha < 0\} = \{\alpha_1, w_{\alpha_1}(\alpha_2), w_{\alpha_1}w_{\alpha_2}(\alpha_3), \cdots, w_{\alpha_1}\cdots w_{\alpha_{n-1}}(\alpha_n)\}.$$

Therefore, $\langle w_{\alpha_{i-1}}\cdots w_{\alpha_1}\Lambda, \alpha_i^\vee\rangle = \langle\Lambda, (w_{\alpha_1}\cdots w_{\alpha_{i-1}}(\alpha_i))^\vee\rangle$.

Now we compute the intertwining operator in the case of SL_2. Let $G = SL_2(\mathbb{Q}_p)$ and χ_p is a character of $\mathbb{Q}_p^\times$. Let

$$A(s,\chi_p)f_p(g) = \int_{N_p} f_p(w_0^{-1}ng)\,dn,$$

be the intertwining operator for $I(s,\chi_p)$, where $N_p = \{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\}$, and $w_0 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

Theorem 6.3 (Computation in the case of $SL_2(\mathbb{Q}_p)$; Gindikin-Karpelevich formula). *Let χ_p be an unramified character. Then*

$$A(s,\chi_p)f_p^0(e) = \frac{1-\chi_p(p)p^{-s-1}}{1-\chi_p(p)p^{-s}} = \frac{L(s,\chi_p)}{L(s+1,\chi_p)},$$

where f_p^0 is the spherical function such that $f_p^0(e) = 1$. It is the unique $SL_2(\mathbb{Z}_p)$-fixed function satisfying $f_p^0\left(\begin{pmatrix}a & x\\ 0 & a^{-1}\end{pmatrix}g\right) = \chi_p(a)|a|^{s+1}f_p^0(g)$.

Proof. We compute $A(s,\chi_p)f_p^0(e) = \int_{\mathbb{Q}_p} f_p^0(w_0^{-1}n)\,dx$. The Haar measure dx satisfies; $\int_{\mathbb{Q}_p} f(xy)\,dy = |x|^{-1}\int_{\mathbb{Q}_p} f(y)\,dy$, and $\int_{\mathbb{Z}_p} dx = 1$. Hence $\int_{\mathbb{Z}_p^\times} dx = \int_{\mathbb{Z}_p} dx - \int_{p\mathbb{Z}_p} dx = 1 - \frac{1}{p} = \frac{p-1}{p}$. Also $\int_{p^{-m}\mathbb{Z}_p^\times} dx = p^m\int_{\mathbb{Z}_p^\times} dx$.

Any element of $x \in \mathbb{Q}_p$ can be written as $x = p^{-m}u$, $u \in \mathbb{Z}_p^\times$ (units in $\mathbb{Z}_p$), and $m \in \mathbb{Z}$. Note that for $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$,

$$w_0^{-1}n = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

Hence if $x \in \mathbb{Z}_p$, $f_p^0(w_0^{-1}n) = 1$. If $x = p^{-m}u$ with $m \geq 1$, then $f_p^0(w_0^{-1}n) = \chi_p(p^m)(p^{-m})^{s+1}$. Therefore,

$$\begin{aligned} \int_{\mathbb{Q}_p} f_p^0(w_0^{-1}n)\,dx &= \int_{\mathbb{Z}_p} f_p^0(w_0^{-1}n)\,dx + \sum_{m=1}^{\infty} \int_{p^{-m}\mathbb{Z}_p^\times} f_p^0(w_0^{-1}n)\,dx \\ &= 1 + \sum_{m=1}^{\infty} \chi_p(p^m)(p^{-m})^{s+1}p^m(1-\frac{1}{p}) \\ &= 1 + (1-\frac{1}{p})\sum_{m=1}^{\infty}(\chi_p(p)p^{-s})^m = 1 + (1-\frac{1}{p})\frac{\chi_p(p)p^{-s}}{1-\chi_p(p)p^{-s}} \\ &= \frac{1-\chi_p(p)p^{-s-1}}{1-\chi_p(p)p^{-s}}. \end{aligned}$$

More generally over an arbitrary algebraic number field, we can show

$$\int_{F_v} f_v^0(w_0^{-1}n)\,dn = \frac{L(s,\chi_v)}{L(s+1,\chi_v)}.$$

where $L(s,\chi_v) = (1-\chi_v(\varpi_v)q_v^{-1})^{-1}$, ϖ_v is a uniformizer in F_v, and q_v is the number of elements in $\mathcal{O}_v/\mathfrak{p}_v$.

Corollary 6.4.

$$A(\Lambda,\chi_v,w)f_v^0(e) = \prod_{\beta>0,w\beta<0} \frac{L(\langle\Lambda,\beta^\vee\rangle,\chi_v\circ\beta^\vee)}{L(1+\langle\Lambda,\beta^\vee\rangle,\chi_v\circ\beta^\vee)}.$$

If $F = \mathbb{R}$, we leave it as an exercise to show that

$$A(s,1)f_\infty^0(e) = \sqrt{\pi}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})},$$

where f_∞^0 is the spherical function satisfying $f_\infty^0\left(\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} g\right) = |a|^{s+1}f_\infty^0(g)$. Use the fact that

$$w_0^{-1}n = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} = \begin{pmatrix} \Delta_x^{-1} & -x\Delta_x^{-1} \\ 0 & \Delta_x \end{pmatrix} \kappa_{\theta(x)},$$

where $\Delta_x = \sqrt{1+x^2}$, $\theta(x) = \arctan(-\frac{1}{x})$, and $\kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Hence

$$A(s,1)f_\infty^0(e) = \int_{-\infty}^{\infty} (1+x^2)^{-\frac{s+1}{2}}\, dx.$$

Proof of Theorem 6.2. Let $N(w) = U \cap w\overline{U}w^{-1}$, where $\overline{U}$ is the opposite unipotent radical. By the commutator relation, each element of $N(w)$ can be expressed in the form $\prod_{\alpha>0, w^{-1}\alpha<0} e_\alpha(u_\alpha)$, where $u_\alpha \in F$ (in any order).

Some properties of $h_\alpha, e_\alpha, w_\alpha$;

$$\begin{aligned} e_\alpha(t_1)e_\alpha(t_2) &= e_\alpha(t_1+t_2) \\ [e_\alpha(t), e_\beta(u)] &= \begin{cases} 1, & \text{if } \alpha+\beta \notin \Phi \\ e_{\alpha+\beta}(c_{\alpha\beta}tu), & \text{if } \alpha+\beta \in \Phi \end{cases} \\ h_\alpha(t)h_\alpha(u) &= h_\alpha(tu) \\ h_\alpha(t)e_\beta(u)h_\alpha(t)^{-1} &= e_\beta(t^{\langle\beta,\alpha^\vee\rangle}u) \\ w_\alpha e_\beta(u) w_\alpha^{-1} &= e_{w_\alpha\beta}(u). \end{aligned}$$

Consider

$$A(\Lambda,\chi,w)f(g) = \int_{N(w)} f(w^{-1}ng)\, dn = \int_{N(w)} f(w^{-1}nww^{-1}g)\, dn = \int_{\overline{N}(w)} f(nw^{-1}g)\, dn,$$

where $\overline{N}(w) = w^{-1}Uw \cap \overline{U}$. Here each element of $\overline{N}(w)$ can be expressed in the form $\prod_{\alpha>0, w\alpha<0} e_{-\alpha}(u_\alpha)$, where $u_\alpha \in F$ (in any order). Let $w = w_{\alpha_n}w'$. Then

$$\{\alpha>0, w\alpha<0\} = \{w'^{-1}(\alpha_n)\} \cup \{\beta>0, w'\gamma<0\},$$

and $\overline{N}(w) = \overline{N}(w') \cdot U_{-w'^{-1}(\alpha_n)}$. Hence

$$A(\Lambda,\chi,w)f(g) = \int_{U_{-w'^{-1}(\alpha_n)}} \int_{\overline{N}(w')} f(n'e_{-w'^{-1}(\alpha_n)}(u)w'^{-1}w_{\alpha_n}g)\, dn'du.$$

Here $n'e_{-w'^{-1}(\alpha_n)}(u)w'^{-1} = n'w'^{-1}w'e_{-w'^{-1}(\alpha_n)}(u)w'^{-1} = n'w'^{-1}e_{-\alpha_n}(u)$. Hence

$$\begin{aligned} A(\Lambda,\chi,w)f(g) &= \int_{U_{-\alpha_n}} \int_{\overline{N}(w')} f(n'w'^{-1}e_{-\alpha_n}(u)w_{\alpha_n}g)\, dn'du \\ &= \int_{U_{-\alpha_n}} A(\Lambda,\chi,w')f(e_{-\alpha_n}(u)w_{\alpha_n}g)\, du = A(w'\Lambda, w'\chi, w_{\alpha_n})A(\Lambda,\chi,w')f(g). \end{aligned}$$

The above cocycle relation of intertwining operators is quite general. Suppose $\mathbf{P} = P_\theta$, $\theta \subset \Delta$. Let $A(\Lambda,\pi,\theta,w)$ be the intertwining operator for $Ind_P^G \pi \otimes exp(\langle\Lambda, H_P(\,)\rangle)$, where $w \in W$ is such that $w\theta \subset \Delta$. We can write $A(\Lambda,\pi,\theta,w)$ as a product of rank-one intertwining operators (i.e., intertwining operators for maximal parabolic subgroups).

Theorem 6.5. *There exists a finite sequence $\alpha_1, ..., \alpha_n \in \Delta$ such that $\Omega_i = \theta_i \cup \{\alpha_i\}$, $\theta_{i+1} = w_i\theta_i$, where $w_i(\theta_i) \subset \Omega_i$, $w_i(\alpha_i) < 0$. Then $w = w_n \cdots w_1$, and*

$$A(\Lambda, \pi, \theta, w) = A(w_{n-1}\cdots w_1\Lambda, w_{n-1}\cdots w_1\pi, \theta_n, w_n)\cdots A(\Lambda, \pi, \theta, w_1).$$

Here let $\Phi(A_\theta, G)$ be the set of roots of G with respect to A_θ, and $\Phi_r(A_\theta, G)$ be the set of reduced roots. Let $\Phi_r^+(\theta, w) = \{\alpha \in \Phi_r^+(A_\theta, G) : w\alpha < 0\}$. Then

$$\Phi_r^+(\theta, w) = \{\alpha_1, w_1^{-1}\alpha_2, (w_2w_1)^{-1}\alpha_3, ..., (w_{n-1}\cdots w_1)^{-1}\alpha_n\}.$$

We explain the theorem using one example. Suppose $\mathbf{P} = \mathbf{MN} \subset Sp(2n)$, where $\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_k} \times Sp(2l)$, and $\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma$ be a representation of $\mathbf{M}(F)$. Let

$$Ind_P^G \pi_1|det|^{s_1} \otimes \cdots \otimes \pi_k|det|^{s_k} \otimes \sigma,$$

be the induced representation. Then $A(\Lambda, \pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma, w)$ is a product of $A(s_i, \pi_i \otimes \sigma)$, $i = 1, ..., k$, $A(s_i - s_j, \pi_i \otimes \pi_j), A(s_i + s_j, \pi_i \otimes \tilde{\pi}_j)$, $1 \leq i < j \leq k$, where

$A(s_i, \pi_i \otimes \sigma)$ is the rank-one operator for $Ind\, \pi_i|det|^{s_i} \otimes \sigma$, $GL_{n_i} \times Sp(2l) \subset Sp(2(n_i + l))$

$A(s_i - s_j, \pi_i \otimes \pi_j)$ is the rank-one operator for $Ind\, \pi_i|det|^{s_i} \otimes \pi_j|det|^{s_j}$, $GL_{n_i} \times GL_{n_j} \subset GL_{n_i+n_j}$

$A(s_i + s_j, \pi_i \otimes \tilde{\pi}_j)$ is the rank-one operator for $Ind\, \pi_i|det|^{s_i} \otimes \tilde{\pi}_j|det|^{-s_j}$, $GL_{n_i} \times GL_{n_j} \subset GL_{n_i+n_j}$

We return to spherical representations and maximal parabolic case. Let π_v be a spherical representation of $\mathbf{M}(F_v)$, $\mathbf{P} = \mathbf{MN}$ is a maximal parabolic subgroup. Let χ_v be the inducing unramified character. Then

$$A(s\tilde{\alpha}, \pi_v, w_0)f_v^0(e) = \prod_{\beta>0, w_0\beta<0} \frac{L(s\langle\tilde{\alpha}, \beta^\vee\rangle, \chi_v \circ \beta^\vee)}{L(1 + s\langle\tilde{\alpha}, \beta^\vee\rangle, \chi_v \circ \beta^\vee)}.$$

Langlands observed that $\langle\tilde{\alpha}, \beta^\vee\rangle = i$, $i = 1, ..., m$: Let V_i be the subspace of ${}^L\mathfrak{n}$, generated by $E_{\beta^\vee} \in {}^L\mathfrak{n}$, $\langle\tilde{\alpha}, \beta^\vee\rangle = i$, where ${}^L\mathfrak{n}$ is the Lie algebra of ${}^L\mathbf{N}$. For each i, the adjoint action of ${}^LM$ leaves V_i stable. Let r be the adjoint representation of ${}^LM$ on ${}^L\mathfrak{n}$, and $r_i = r|_{V_i}$.

Theorem 6.6 (Langlands-Shahidi). *r_i is irreducible for each i, and the weights of r_i are the roots $\beta^\vee$ in ${}^L\mathfrak{n}$ which restricts to $i\alpha^\vee$ in ${}^LA$.*

Let $r = \oplus_{i=1}^m r_i$. Let t_v be the semi-simple conjugacy class in ${}^LT$. Recall that $\chi_v \circ \beta^\vee(\varpi_v) = \beta^\vee(t_v)$.

Theorem 6.7 (Langlands-Shahidi).

$$A(s\tilde{\alpha}, \pi_v, w_0)f_v^0(e) = \prod_{i=1}^m \frac{L(is, \pi_v, r_i)}{L(1 + is, \pi_v, r_i)},$$

where

$$L(s,\pi_v,r_i)=\prod_{\beta>0,\langle\tilde\alpha,\beta^\vee\rangle=i} L(s,\chi_v\circ\beta^\vee),\quad L(s,\chi_v\circ\beta^\vee)=(1-\chi_v\circ\beta^\vee(\varpi_v)q_v^{-s})^{-1}.$$

Examples 6.8. (1) $\mathbf{G}=GL_{k+l}$, and $\mathbf{M}\simeq GL_k\times GL_l$, $\mathbf{N}=\begin{pmatrix} I_k & X\\ 0 & I_l\end{pmatrix}$. Then ${}^LM\simeq GL_k(\mathbb{C})\times GL_l(\mathbb{C})$, and $r(diag(g_1,g_2))X=g_1Xg_2^{-1}$. Here r is irreducible. So in this case $m=1$. More explicitly, let $\pi_1=\pi(\mu_1,...,\mu_k)$, $\pi_2=\pi(\nu_1,...,\nu_l)$ be spherical representations of $GL_k(F),GL_l(F)$, resp. Then

$$L(s,\pi_1\otimes\pi_2,r)=L(s,\pi_1\times\tilde\pi_2)=\prod_{i=1}^k\prod_{j=1}^l L(s,\mu_i\nu_j^{-1}).$$

(2) $\mathbf{G}=Sp(2n)$, $\mathbf{M}=GL_n$ (Siegel parabolic subgroup). Then $\mathbf{N}$ is generated by $\{e_i+e_j, i<j, 2e_i, i=1,...,n\}$, and $\tilde\alpha=e_1+\cdots+e_n$. We can see that $m=2$; $r=r_1\oplus r_2$; ${}^L\mathfrak{n}=V_1\oplus V_2$, where V_1 is spanned by $e_i, i=1,...,n$ (note that e_i is the coroot of $2e_i$); V_2 is spanned by $e_i+e_j, i<j$. Hence, for a spherical representation π of $GL_n(F)$,

$$L(s,\pi,r_1)=L(s,\pi),\text{ standard } L\text{-function of } GL_n$$
$$L(s,\pi,r_2)=L(s,\pi,\wedge^2),\text{ exterior square } L\text{-function of } GL_n$$

(3) Consider E_7-1 case in Example 1.36. We saw that there is a rational injection $f:\mathbf{M}\longrightarrow GL_2\times GL_3\times GL_4$. Then f induces an injection $f:\mathbf{M}(\mathbb{A})\longrightarrow GL_2(\mathbb{A})\times GL_3(\mathbb{A})\times GL_4(\mathbb{A})$ such that $\mathbf{M}(\mathbb{A})(\mathbb{A}^*)^2$ is co-compact in $GL_2(\mathbb{A})\times GL_3(\mathbb{A})\times GL_4(\mathbb{A})$, where $(\mathbb{A}^*)^2$ is embedded as a center of the first two factors.

Let π_i, $i=1,2,3$, be cuspidal representations of $GL_{1+i}(\mathbb{A})$ with central characters ω_i, resp. Then $\pi_1\otimes\pi_2\otimes\pi_3|_{f(\mathbf{M}(\mathbb{A}))}$ decomposes as a direct sum of irreducible representations of $\mathbf{M}(\mathbb{A})$. Let π be any cuspidal irreducible constituent of this direct sum. Then $\omega_\pi=\omega_1^6\omega_2^4\omega_3^3$. Let $\pi=\otimes_v\pi_v$. If π_v is a spherical representation, we can see that $m=4$, and

$$L(s,\pi_v,r_1)=L(s,\pi_{1v}\times\pi_{2v}\times\pi_{3v}),$$
$$L(s,\pi_v,r_2)=L(s,\tilde\pi_{2v}\otimes\pi_{3v},(\rho_3\otimes\omega_1\omega_2)\otimes\wedge^2\rho_4),$$
$$L(s,\pi_v,r_3)=L(s,(\pi_{1v}\otimes\omega_1\omega_2\omega_3)\times\tilde\pi_{3v}),$$
$$L(s,\pi_v,r_4)=L(s,\pi_{2v}\otimes\omega_1^2\omega_2\omega_3).$$

List of L-functions via Langlands-Shahidi method (split reductive groups).

A_n case.

$GL_m\times GL_n\subset GL_{m+n}$ gives the Rankin-Selberg L-function $L(s,\pi_1\times\pi_2)$, where $\pi_i's$ are cuspidal representations of GL_m,GL_n, resp.

B_n case.

$B_n - 1$: $m = 2$; r_1 gives the Rankin-Selberg L-function of $GL_k \times GSpin(2l+1)$; r_2 gives the twisted symmetric square L-function of GL_k. If $G = SO(2n+1)$, r_1 gives the Rankin-Selberg L-function of $GL_k \times SO(2l+1)$; r_2 gives the symmetric square L-function of GL_k.

C_n case.

$C_n - 1$: $m = 2$; r_1 gives the Rankin-Selberg L-function of $GL_k \times Sp(2l)$; r_2 gives the exterior square L-function of GL_k, if $k \neq 1$ (If $k = 1$, then $m = 1$).

D_n case.

$D_n - 1$: $m = 2$; r_1 gives the Rankin-Selberg L-function of $GL_k \times GSpin(2l)$; r_2 gives the twisted exterior square L-function of GL_k. If $G = SO(2n)$, r_1 gives the Rankin-Selberg L-function of $GL_k \times SO(2l)$; r_2 gives the exterior square L-function of GL_k (If $k = 1$, then $m = 1$).

$D_n - 2$: $m = 2$; r_1 gives the triple L-function of $GL_{n-2} \times GL_2 \times GL_2$; r_2 gives the twisted exterior square L-function of GL_{n-2}.

$D_n - 3$: $m = 2$; r_1 gives $L(s, \sigma \otimes \tau, \rho_{n-3} \otimes \wedge^2 \rho_4)$; r_2 gives the twisted exterior square L-function of GL_{n-3}.

F_4 case.

$F_4 - 1$: $m = 4$; r_1 gives $L(s, \sigma \times \tau)$ which is entire; r_2 gives $L(s, \sigma \otimes \tau, Sym^2 \rho_2 \otimes \rho_3)$ which has a pole at $s = 1$ when $\tilde{\tau} \simeq Sym^2 \sigma$.

$F_4 - 2$: $m = 3$; r_1 gives $L(s, \sigma \otimes \tau, Sym^2 \rho_3 \otimes \rho_2)$; r_2 gives $L(s, \sigma, Sym^2 \rho_3 \otimes \omega_\tau)$ which has a pole at $s = 1$ always.

$(xviii)$: $m = 2$; $M = GSpin(7) \subset F_4$; $dim\, r_2 = 1$; r_1 is the 14-dim'l irreducible representation of $Sp_6(\mathbb{C})$, called spherical harmonic.

$(xxii)$: $m = 2$; $M = GSp_6 \subset F_4$; r_1=8-dim'l spin representation of $Spin(7, \mathbb{C})$; r_2 gives the standard L-function of $SO_7(\mathbb{C})$ (7-dimensional).

E_6 case.

$E_6 - 1$: $m = 3$; r_1 gives the triple L-function of $GL_3 \times GL_2 \times GL_3$; r_2 gives the standard L-function of $GL_3 \times GL_3$.

$E_6 - 2$: $m = 2$; $r_1 = \wedge^2 \rho_5 \otimes \rho_2$; r_2 gives the Rankin-Selberg L-function of $GL_5 \times GL_2$ which is entire.

(x): $m = 2$; $dim\, r_2 = 1$; r_1 gives the exterior cube L-function of $GL_6(\mathbb{C})$ (20 dim'l irreducible representation of $GL_6(\mathbb{C})$)

$(xxiv)$: $m = 1$; r_1=16-dimensional half-spin representation of $Spin(10)$.

E_7 case.

$E_7 - 1$: $m = 4$; r_1 gives the triple L-function of $GL_3 \times GL_2 \times GL_4$; r_2 comes from $D_6 - 3$ case.

$E_7 - 2$: $m = 3$; $r_1 = \wedge^2 \rho_5 \otimes \rho_3$; r_2 gives the Rankin-Selberg L-function of $GL_5 \times GL_3$ which is entire.

$E_7 - 3$: $m = 2$; r_1 gives the L-function $L(s, \sigma \otimes \tau, Spin^{16} \otimes \rho_2)$, where σ, τ are cuspidal representations of $GSpin_{10}, GL_2$, resp. and $Spin^{16}$ is the 16-dimensional half-spin representation of $Spin_{16}(\mathbb{C})$.

$E_7 - 4$: $m = 3$; r_1 gives the L-function $L(s, \sigma \otimes \tau, \wedge^2 \rho_6 \otimes \rho_2)$, where σ, τ are cuspidal representations of GL_6, GL_2, resp.

(xi): $m = 2$; r_1 gives the exterior cube L-function of $GL_7(\mathbb{C})$ (35-dimensional representation of $GL_7(\mathbb{C})$); r_2 gives the standard L-function of $GL_7(\mathbb{C})$ which is entire.

$(xxvi)$: $m = 2$; $dim\, r_2 = 1$ and r_1 gives the degree $32 = 2^5$ spin L-function of $Spin_{12}$

(xxx): $m = 1$; r_1 gives the standard L-function of E_6.

E_8 case.

$E_8 - 1$: $m = 6$; r_1 gives the triple L-function of $GL_3 \times GL_2 \times GL_5$; r_2 comes from $E_7 - 2$ case.

$E_8 - 2$: $m = 5$; r_1 gives the L-function $L(s, \sigma \otimes \tau, \wedge^2 \rho_5 \otimes \rho_4)$, where σ, τ are cuspidal representations of GL_5, GL_4, resp.

$E_8 - 3$: $m = 4$; r_1 gives the L-function $L(s, \sigma \otimes \tau, Spin^{16} \otimes \rho_3)$, where σ, τ are cuspidal representations of $GSpin_{10}, GL_3$, resp. and $Spin^{16}$ is the16-dimensional half-spin representation of $Spin_{16}(\mathbb{C})$.

$E_8 - 4$: $m = 3$; r_1 gives the standard L-function of $E_6 \times GL_2$; r_2 gives the standard L-function of E_6 ((xxx) case).

$E_8 - 5$: $m = 4$; r_1 gives the L-function $L(s, \sigma \otimes \tau, \wedge^2 \rho_7 \otimes \rho_2)$, where σ, τ are cuspidal representations of GL_7, GL_2, resp.

$(xiii)$: $m = 3$; r_1 gives the degree 56 exterior cube L-function of GL_8

$(xxviii)$: $m = 2$; r_1 gives the degree $64 = 2^6$ spin L-function of $Spin_{14}$

$(xxxii)$: $m = 2$; $dim\, r_2 = 1$; r_1 gives the standard L-function of E_7.

G_2 case.

(xv) (attached to the maximal parabolic subgroup generated whose unipotent radical contains the long simple root subgroup) : $m = 2$; $dim\, r_2 = 1$, r_1 gives the third symmetric power L-function of GL_2

(xvi) (attached to the maximal parabolic subgroup generated whose unipotent radical contains the short simple root subgroup): $m = 3$; $dim\, r_2 = 1$, r_1 gives the standard L-function of GL_2