3. L-groups and automorphic L-functions.

Suppose $\mathbf{G}$ is reductive and $\mathbf{T}$ is a maximal torus. Recall the root datum $(X^*(\mathbf{T}), \Phi, X_*(\mathbf{T}), \Phi^\vee)$ of $\mathbf{G}$.

Definition 3.1. *The complex group whose root datum is the dual root datum $(X_*(\mathbf{T}), \Phi^\vee, X^*(\mathbf{T}), \Phi)$, is called the L-group of $\mathbf{G}$, and is denoted by ${}^LG$. Since $\mathbf{G}$ is split, we do not consider the action of $Gal(\bar{F}/F)$.*

Suppose $\mathbf{G}$ is semi-simple. Recall that $\mathbf{G}$ is simply connected if $X_*(\mathbf{T}) = \mathbb{Z}$-span of $\Phi^\vee$, and $\mathbf{G}$ is adjoint if $X^*(\mathbf{T}) = \mathbb{Z}$-span of Φ. Hence $\mathbf{G}$ is simply connected if and only if ${}^LG$ is adjoint; $\mathbf{G}$ is adjoint if and only if ${}^LG$ is simply connected.

Since the exceptional groups of type F_4, G_2 and E_8 are both simply connected and adjoint, their L-groups are just complex group $\mathbf{G}(\mathbb{C})$.

Example 3.2. If $\mathbf{G} = SL_n$, then ${}^LG = PGL_n(\mathbb{C})$; if $\mathbf{G} = SO(2n+1)$, then ${}^LG = Sp(2n, \mathbb{C})$. If $\mathbf{G} = Sp(2n)$, then ${}^LG = SO(2n+1, \mathbb{C})$. If $\mathbf{G} = SO(2n)$, then ${}^LG = SO(2n, \mathbb{C})$. If $\mathbf{G}$ is the simply connected group of type E_6, E_7, then ${}^LG$ is the adjoint group of type E_6, E_7.

Suppose $\mathbf{G}$ is reductive. Then calculation of ${}^LG$ is a little more complicated. We use

Lemma 3.3 (Borel). *The derived group of ${}^LG$ is simply connected if and only if the center of $\mathbf{G}$ is connected.*

Examples 3.4. (1) If $\mathbf{G} = GL_n$, then ${}^LG = GL_n(\mathbb{C})$.

(2) Let $\mathbf{G} = GSp(2n)$. The center of $\mathbf{G}$ is connected. Hence the derived group of ${}^LG$ is the simply connected group of type B_n, namely, $Spin(2n+1)$. Then we can show that ${}^LG = GSpin(2n+1, \mathbb{C})$. If $n = 2$, by accidental isomorphism, $Spin(5) \simeq Sp(4)$. Hence ${}^LGSp(4) = GSp(4, \mathbb{C})$.

(3) Let $\mathbf{G}$ be the simply connected group of type E_6, and let $\theta = \Delta - \{\alpha_1\}$. Then $P_\theta = \mathbf{MN}$. Here we can show that $\mathbf{M}_D$ (the derived group of $\mathbf{M}$) $\simeq Spin(10)$ and there is a surjective map $GSpin(10) \longrightarrow \mathbf{M}$, with the kernel $\{\pm 1\}$. Hence we have a dual map ${}^LM \longrightarrow GSO(10, \mathbb{C}) = {}^LGSpin(10)$. Since the center of $\mathbf{M}$ is connected, the derived group of ${}^LM$ is simply connected. Hence it is $Spin(10, \mathbb{C})$. Therefore ${}^LM = GSpin(10, \mathbb{C})$.

Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $\mathbf{G}$ and $r : {}^LG \longrightarrow GL_N(\mathbb{C})$ be a finite-dimensional representation. For almost all v ($v \notin S$; S is a finite set of places, including all archimedean places), π_v is spherical. So by Theorem 2.4, π_v is uniquely determined by a semi-simple conjugacy class $\{t_v\} \subset {}^LT$. We form a local Langlands' L-function

$$L(s, \pi_v, r) = det(I - r(t_v) q_v^{-s})^{-1}.$$

Example 3.5. Let $\pi = \otimes \pi_v$ be a cuspidal representation of GL_2. Let $diag(\alpha_v, \beta_v)$ be a semi-simple conjugacy class of π_v. Let $Sym^m : GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ be the m-th symmetric power represenation. Then

$$L(s, \pi_v, Sym^m) = \prod_{i=0}^{m} (1 - \alpha_v^{m-i} \beta_v^i q_v^{-s})^{-1}.$$

Theorem 3.6 (Langlands). *Let $L_S(s,\pi,r)=\prod_{v\notin S}L(s,\pi_v,r)$ be a partial L-function. Then $L_S(s,\pi,r)$ converges absolutely for $Re(s)>>0$ and defines a holomorphic function there.*

Conjecture (Langlands). *$L_S(s,\pi,r)$ has a meromorphic continuation to all of $\mathbb{C}$.*

This is not so obvious. For example, let $L(s)=\prod_{p\equiv 1(\mathrm{mod} 4)}(1-p^{-s})^{-1}$. We know that $L(s)$ is absolutely convergent for $Re(s)>1$. However, it has been shown that $L(s)$ has a natural boundary at $Re(s)=0$.

In order to formulate more refined conjecture, let $\psi=\otimes\psi_v$ be a character of $\mathbb{A}/F$. It is necessary to define γ and ϵ-factors.

Conjecture (Langlands). *For each $v\in S$, we can define a local L-function $L(s,\pi_v,r)$ (of the form $P(q_v^{-s})^{-1}$, where $P(X)$ is a polynomial with the constant term 1), and local root number $\epsilon(s,\pi_v,r,\psi_v)$ (of the form Aq_v^{-Bs}), such that $L(s,\pi,r)=\prod_{\text{all } v}L(s,\pi_v,r)$ has a meromorphic continuation to all of $\mathbb{C}$, and satisfies a functional equation $L(s,\pi,r)=\epsilon(s,\pi,r)L(1-s,\pi,\tilde{r})$, where $\epsilon(s,\pi,r)=\prod_v\epsilon(s,\pi_v,r,\psi_v)$, and $\tilde{r}$ is the contragredient representation defined by $\tilde{r}(g)={}^tr(g)^{-1}$.*

We call $L(s,\pi,r)$ automorphic L-functions. There are two ways of studying automorphic L-functions:

(1) Method of Rankin-Selberg; integral representations;

We can express many automorphic L-functions as integrals of Eisenstein series, theta functions, etc. The simplest example is Riemann zeta function: $\zeta(s)=\prod_p(1-p^{-s})^{-1}$. Let $\xi(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. Then we have an integral representation

$$\xi(s)=\int_0^\infty x^{\frac{s}{2}}g(x)\,\frac{dx}{x},$$

where $g(x)=\frac{1}{2}(\theta(x)-1)$, and $\theta(x)=\sum_{n=-\infty}^{\infty}e^{-n^2\pi x}$. The properties of the theta function provide the meromorphic continuation and the functional equation of $\xi(s)$, namely, $\xi(s)$ is holomorphic except at $s=0,1$ and $\xi(1-s)=\xi(s)$.

However, in many cases, we can only express $L(s,\pi,r)\times$ (archimedean factor) as an integral. The integral behaves well but in order to conclude that $L(s,\pi,r)$ is entire, one needs to show that the archimedean factor is non-vanishing. This has been very difficult to show that.

(2) Langlands-Shahidi method:

Langlands observed that many L-functions appear in the constant terms of Eisenstein series associated to cuspidal representations of the Levi subgroups of maximal parabolic subgroups of Chevalley groups. The meromorphic continuation and the functional equation of Eisenstein series provide the analytic properties of the L-functions. Langlands proved the meromorphic continuation (but no functional equation). Shahidi extended Langlands' observation to quasi-split groups, and calculated non-constant terms and thus obtained the functional equations.