2. Cuspidal representations.

In this section, let $\mathbf{G}$ be a split reductive group defined over a number field F with the ring of adeles $\mathbb{A}$. Let Z be the center of $\mathbf{G}$. Let ω be a grössencharacter of F, and let $L^2(\omega) = L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}), \omega)$ be the Hilbert space of square integrable functions modulo the center, i.e.,

$$\int_{Z(\mathbb{A})\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})} |\phi(g)|^2 \, dg < \infty,$$

such that $\phi(zg) = \omega(z)\phi(g)$ for $z \in Z(\mathbb{Z})$ and $g \in \mathbf{G}(\mathbb{A})$. Denote by R, the right regular representation of $\mathbf{G}(\mathbb{A})$ on $L^2(\omega)$, i.e., $(R(h)\phi)(g) = \phi(gh)$ for $g, h \in \mathbf{G}(\mathbb{A})$ and $\phi \in L^2$.

Let $L_0^2(\omega) = L_0^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}), \omega)$ be the subspace of cuspidal functions, i.e.,

$$\int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} \phi(ng) \, dn = 0$$

for all unipotent radical $\mathbf{N}$. (It is enough to consider the unipotent radicals of maximal parabolic subgroups.) Clearly, R is a unitary representation of $\mathbf{G}(\mathbb{A})$ on $L_0^2(\omega)$.

Theorem 2.1 (Gelfand, Piatetski-Shapiro for number field case, Harder for function field case). *R is a direct sum of irreducible unitary representations of $\mathbf{G}(\mathbb{A})$. i.e.,*

$$R = \oplus\, m(\pi)\pi.$$

We call the irreducible constituents cuspidal representations of $\mathbf{G}(\mathbb{A})$.

Theorem 2.2 (Piatetski-Shapiro, Shalika). *(Multiplicity one result for GL_n) If $\mathbf{G} = GL_n$, then $m(\pi) = 1$ for all cuspidal representations.*

Theorem 2.3 (Gelfand, Piatetski-Shapiro, Flath). *Let π be a cuspidal representation of $\mathbf{G}(\mathbb{A})$. Then there is a non-unique decomposition*

$$\pi = \otimes_v \pi_v,$$

where π_v is an irreducible unitary representation of $\mathbf{G}(F_v)$ for all v, and for almost all $v < \infty$, π_v has a vector fixed by the action of $\mathbf{G}(\mathcal{O}_v)$. We call such π_v spherical representation or class one representation. It is the choices made for these vectors which lead to the decomposition $\pi = \otimes \pi_v$ in a non-unique way. However the equivalence classes of π_v are all unique.

Theorem 2.4. *A spherical representation π_v is uniquely determined by its Hecke conjugacy class $\{t_v\} \subset {}^L G$. (the L-group of $\mathbf{G}$). We will explain this later using the principal series.*

Example 2.5. If $\mathbf{G} = GL_n$, then ${}^L G = GL_n(\mathbb{C})$ and a spherical representation π_v is determined by a semi-simple conjugacy class $\{diag(\alpha_1, ..., \alpha_n)\}$.

Theorem 2.6 (Piatetski-Shapiro, Jacquet-Shalika). *(Strong multiplicity one result for GL_n) Suppose $\pi = \otimes\pi_v$ and $\pi' = \otimes\pi'_v$ are two cuspidal representations of GL_n such that $\pi_v \simeq \pi'_v$ for almost all v. Then $\pi \simeq \pi'$. Hence by multiplicity one result, $\pi = \pi'$.*

Examples 2.7. If $\mathbf{G} = GL_2$, and $F = \mathbb{Q}$, there are two classes of cuspidal representations.

(1) cuspidal representations attached to holomorphic Hecke eigenforms: $\pi = \pi_f$, where f is a classical holomorphic cusp form of weight k with respect to a congruence subgroup Γ of $SL_2(\mathbb{Z})$, which is an eigenform of all Hecke operators. Namely, f is a holomorphic function on the upper half plane $\{z = x + iy, y > 0\}$ and

$$f(\gamma \cdot z) = \chi(a)(cz+d)^k f(z), \quad \gamma \cdot z = \frac{az+b}{cz+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

where $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} :, c \equiv 0(N)\}$ and χ is a character of $(\mathbb{Z}/N\mathbb{Z})^*$. Also f vanishes on every cusp of $\Gamma_0(N)$. Recall the strong approximation:

$$GL_2(\mathbb{A}_{\mathbb{Q}}) = GL_2(\mathbb{Q}) GL_2^+(\mathbb{R}) K_0(N),$$

where $K_0(N) = \prod_{p \nmid N} GL_2(\mathbb{Z}_p) \prod_{p|N} K_{p,N}$, and $K_{p,N} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) : c \equiv 0(N)\}$. Moreover, $GL_2(\mathbb{Q}) \cap GL_2^+(\mathbb{R}) K_0(N) = \Gamma_0(N)$.

For $g \in GL_2^+(\mathbb{R})$, define $j(g,z) = (cz+d)(det g)^{-\frac{1}{2}}$. The character χ defines a character of $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times$, and we write $\chi = \otimes\chi_p$. Each χ_p defines a character of $K_{p,N}$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \chi_p(a)$. Hence we obtain a character of $K_0(N)$. Now define

$$\phi_f(\gamma g_\infty k_0) = f(g_\infty \cdot i) j(g_\infty, i)^{-k} \chi(k_0),$$

for $\gamma \in GL_2(\mathbb{Q}), g_\infty \in GL_2^+(\mathbb{R})$, and $k_0 \in K_0(N)$. Then

$$\phi_f \in L_0^2(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_{\mathbb{Q}}), \chi).$$

We can show that the subspace generated by ϕ_f is irreducible if and only if f is an eigenform of all Hecke operators. We have a Fourier expansion: $f(z) = \sum_{n=1}^\infty a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}$, $a_f(1) = 1$. Suppose $\pi = \pi_f = \otimes_p \pi_p$. Then π_p is spherical for $p \nmid N$ and if $diag(\alpha_p, \beta_p)$ is the semi-simple conjugacy class of π_p, then $a_f(p) = \alpha_p + \beta_p$.

(2) cuspidal representations attached to Maass cusp forms: $\pi = \pi_f$, where f is an eigenfunction of the Laplace-Beltrami operator $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$, and also eigenforms for all Hecke operators. Let $\Delta f = \frac{1}{4}(1-s^2) f$. Then $s \in i\mathbb{R}$, or $s \in \mathbb{R}$ and $|s| < 1$. We have also Fourier expansion: $f(z) = \sum_{n \neq 0} a_f(n) |n|^{-\frac{1}{2}} W(nz)$, where $W(z) = \sqrt{y} K_s(2\pi y) e^{2\pi i x}$, and K_s is a Bessel function. Again let $\pi = \pi_f = \otimes_p \pi_p$. If $diag(\alpha_p, \beta_p)$ is the semi-simple conjugacy class of π_p, then $a_f(p) = \alpha_p + \beta_p$.

Ramanujan Conjecture. $|\alpha_p| = |\beta_p| = 1$.

Selberg Conjecture. $s \in i\mathbb{R}$. *Hence the eigenvalues* $\frac{1}{4}(1 - s^2) \geq \frac{1}{4}$.