# AUTOMORPHIC $L$-FUNCTIONS

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## 0. Introduction.

The goal of this course is to give a proof of functoriality of symmetric cube and symmetric fourth of cuspidal representations of $GL_2(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of a number field $F$. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$. Let $Sym^m : GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ be the symmetric $m$th power representation. By the local Langlands' correspondence, $Sym^m(\pi_v)$ is a well-defined irreducible admissible representation of $GL_{m+1}(F_v)$ for each $v$. Then $Sym^m(\pi) = \otimes_v Sym^m(\pi_v)$ is an irreducible admissible representation of $GL_{m+1}(\mathbb{A})$.

**Conjecture 0.1 (Langlands).** *$Sym^m(\pi)$ is an automorphic representation.*

**Theorem 0.2.**

(1) *(Gelbart-Jacquet) $Sym^2(\pi)$ is an automorphic representation of $GL_3(\mathbb{A}_F)$.*
(2) *(Kim-Shahidi) $Sym^3(\pi)$ is an automorphic representation of $GL_4(\mathbb{A}_F)$.*
(3) *(Kim) $Sym^4(\pi)$ is an automorphic representation of $GL_5(\mathbb{A}_F)$.*

We use the Langlangs-Shahidi method and the converse theorem of Cogdell-Piatetski-Shapiro. For this, we need exceptional groups of type $E_6, E_7$ and $D_{2n}$ (spin groups). We will first develop necessary background. The following is a syllabus for the course.

(1) Chevalley groups and their properties
(2) Cuspidal representations
(3) $L$-groups and automorphic $L$-functions
(4) Induced representations and intertwining operators
(5) Eisenstein series and constant terms
(6) $L$-functions in the constant terms
(7) Meromorphic continuation of $L$-functions
(8) Generic representations and their Whittaker models
(9) Local coefficients and non-constant terms
(10) Local Langlands conjecture
(11) Local $L$-functions and functional equations
(12) Normalization of intertwining operators

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(13) Langlands' functoriality
(14) Converse theorem of Cogdell and Piatetski-Shapiro
(15) Holomorphy and boundedness in vertical strips of $L$-functions
(16) Functoriality of symmetric cube
(17) Functoriality of symmetric fourth

## 1. Chevalley groups and their properties.

This section is mostly from [Bo], [Hum], [Sa], and [Se].

*1.1 Algebraic groups.* Let $F$ be a field, and let $\mathbf{G}$ be an affine algebraic group defined over $F$. The easiest way to define it, is to define it as a subgroup of $GL(n,\Omega)$, where $\Omega$ is an algebraically closed field, containing $F$ such that $\mathbf{G}$ is the set of solutions of a finite set of polynomial equations in $X_{ij}$ with coefficients in $F$ (we assume, for simplicity, that $F$ is of characteristic zero). In algebraic geometry terms, $\mathbf{G}$ is an algebraic variety (closed subset in Zariski topology).

*Examples 1.1.*

(1) $\mathbf{G} = GL(n)$. It is a closed set in $\Omega^{n^2+1}$ defined by a polynomial $det(X_{ij})Y = 1$. We usually denote $GL(1)$ by $\mathbb{G}_m$ and $\Omega$ by $\mathbb{G}_a$, which is isomorphic to the group $\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\}$
(2) $\mathbf{G} = SL(n)$ is defined by $det(X_{ij}) - 1$.
(3) $\mathbf{G} = Sp(2n) = \{g \in SL(2n) \mid {}^t g J g = J,\ J = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix},\ J_n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{pmatrix}\}$

This is given by $4n^2$ polynomial equations. It is called symplectic group. Note that $Sp(2) = SL(2)$.
(4) $\mathbf{G} = R_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{G}_m)$ is called the restriction of scalars, because $\mathbf{G}(\mathbb{Q}) = GL(1, \mathbb{Q}(i))$. It is given by $\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in GL(2,\Omega)\}$. Similarly, $\mathbf{G} = R_{\mathbb{Q}(i)/\mathbb{Q}}(GL(2))$ is defined.

We will always write in bold letters to denote algebraic groups, in order to emphasize that it is a subgroup of $GL(n,\Omega)$. If $\mathbf{G}$ is an algebraic group, then $\mathbf{G}(F)$ denotes the group of $F$-rational points. If $\mathbf{G} \subset GL(n)$ is defined by a set of polynomials with coefficients in a commutative ring $R$, we can define the group of $R$-rational points by $\mathbf{G}(R) = \mathbf{G} \cap R^{n^2+1}$. Hence, for example, $GL(n,R)$ is the set of $n \times n$ matrices over $R$ whose determinants are units in $R$. So $GL(2,\mathbb{Z})$ is the set of $2 \times 2$ matrices with integer components and whose determinants are $\pm 1$.

**Definition 1.2.** *Let $\mathbf{G}$ be a connected algebraic group. The radical of $\mathbf{G}$, denoted by $R(\mathbf{G})$ is a maximal element in the set of connected solvable normal subgroups of $\mathbf{G}$. The unipotent radical of $\mathbf{G}$, denoted by $R_u(\mathbf{G})$, is the unipotent part of $R(\mathbf{G})$, i.e., the set of unipotent elements of $R(\mathbf{G})$.*

Recall that $g \in \mathbf{G}$ is semi-simple if $g$ is similar to a diagonal matrix; $g$ is unipotent if $(g-1)^m = 0$ for some positive integer $m$. By Jordan decomposition, any $g$ can be written uniquely $g = g_s g_u$, where $g_s$ is semi-simple, $g_u$ unipotent and $g_s, g_u$ commute.

**Definition 1.3.** *$\mathbf{G}$ is called semi-simple if $R(\mathbf{G}) = 1$; $\mathbf{G}$ is called reductive if $R_u(\mathbf{G}) = 1$.*

One can think of reductive groups as groups like $GL(n)$. Semi-simple groups are like $SL(n)$, where the center is finite.

**Proposition 1.4.** *(1) (Levi decomposition) Suppose $\mathbf{G}$ is a connected algebraic group defined over a field $F$ of characteristic zero. Then there exists a reductive subgroup $\mathbf{M} \subset \mathbf{G}$ such that $\mathbf{G} = \mathbf{M}R_u(\mathbf{G})$ (semi-direct product).*

*(2) Suppose $\mathbf{G}$ is reductive. Then $\mathbf{G} = R(\mathbf{G}) \cdot \mathbf{G}'$ (almost direct product, i.e., the intersection is finite), where $\mathbf{G}'$ is the derived group, i.e., $\mathbf{G}' = [\mathbf{G}, \mathbf{G}]$. Also $R(\mathbf{G})$ is the connected component of the center of $\mathbf{G}$.*

We will use the Levi decomposition mostly in the case of parabolic subgroups.

**Definition 1.5.** *An algebraic group $\mathbf{T}$ defined over $F$ is called torus if $\mathbf{T}$ is isomorphic to $GL(1)^n$ for some $n \in \mathbb{Z}_+$. A torus $\mathbf{T}$ is called split over $F$ if the isomorphism is defined by a polynomial with coefficients in $F$.*

*Examples 1.6.* Let $D(n)$ be the group of diagonal matrices in $GL(n)$. Then clearly, $D(n)$ is a torus. The algebraic group $\mathbf{G} = R_{\mathbb{Q}(i)/\mathbb{Q}} GL(1)$ in examples 1.1, is a torus. In fact, $\mathbf{G} \simeq GL(1)^2$. The isomorphism is $\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \longmapsto \begin{pmatrix} x+yi & 0 \\ 0 & x-yi \end{pmatrix}$. Notice that the isomorphism is not defined by polynomials with coefficients in $\mathbb{Q}$. This is an example of quasi-split torus. In this course, we will only deal with split tori.

**Definition 1.7.** *Let $\mathbf{G}$ be an algebraic group. Define $X^*(\mathbf{G})$ to be the group of characters defined over $F$, i.e., the group of homomorphisms $\mathbf{G} \longrightarrow GL(1)$, defined by a polynomial.*

**Proposition 1.8.** *$X^*(GL(1)^n) \simeq \mathbb{Z}^n$.*

We can give the characters explicitly; $\chi(x_1, ..., x_n) = x_1^{m_1} \cdots x_n^{m_n}$ for $m_i \in \mathbb{Z}$.

**Proposition 1.9.** *Let $\mathbf{G}$ be reductive and $\mathbf{G} = S \cdot \mathbf{G}'$, where $S = R(\mathbf{G})$ and $\mathbf{G}'$ is the derived group. Then $S$ is a torus and $X*(\mathbf{G})$ is a subgroup of $X*(S)$ with finite index.*

*Proof.* We first show that $X*(\mathbf{G}) = 1$ if $\mathbf{G}$ is semi-simple. If $\mathbf{G}$ is simple (i.e., it has no proper closed normal subgroup of dimension$> 0$), then given $\chi : \mathbf{G} \longrightarrow GL(1)$, $ker\chi$ is a closed normal subgroup. By dimension formula, $dim\mathbf{G} = dim(Im\chi) + dim(ker\chi)$. Hence $dim(ker\chi) \geq 1$. So $ker\chi = \mathbf{G}$. If $\mathbf{G}$ is semi-simple, there is an isogeny (surjective homomorphism with a finite kernel) $\prod_{i=1}^k \mathbf{G}_i \longrightarrow \mathbf{G}$, where $\mathbf{G}_i$ is simple. Then $X^*(\mathbf{G}) \longrightarrow \prod_{i=1}^k X^*(\mathbf{G}_i)$ is an injection. Hence $X^*(\mathbf{G}) =$

1. Suppose $\mathbf{G}$ is reductive. Then there exists an isogeny $S \times \mathbf{G}' \longrightarrow \mathbf{G}$. Then $X^*(\mathbf{G}) \longrightarrow X^*(S) \times X^*(\mathbf{G}') = X^*(S)$ is an injection with finite index. $\square$

*Example 1.10.* Suppose $\mathbf{G} = GL(n) = Z \cdot SL(n)$, where $Z = \{aI_n | \, a \in GL(1)\}$. Then $X^*(Z) =< \chi : aI_n \mapsto a > \simeq \mathbb{Z}$. But $X^*(GL(n)) =< det >$, where $det$ is the character $g \mapsto det(g)$. But $det(aI_n) = a^n$.

*1.2 Roots and coroots.* Let $\mathbf{G}$ be an algebraic group. We can define its Lie algebra, denoted by $\mathfrak{g}$. It is the set of left invariant derivations of the algebra of algebraic functions on $\mathbf{G}$. Rather than defining it abstractly, we show how to find it in the case when $\mathbf{G} \subset GL(n, \Omega)$: Take $t$ such that $t^2 = 0$. Then $\mathfrak{g} = \{X \in M(n, \Omega) | \, 1 + tX \in \mathbf{G}\}$. Note that $\mathfrak{g}$ is a vector space whose dimension is $dim\, \mathbf{G}$.

*Examples 1.11.* (1) $\mathbf{G} = SL(n)$. Then $\mathfrak{g} = \{X \in M(n) | \, det(1 + tX) = 1\} = \{X \in M(n) | \, trX = 0\}$.

(2) $\mathbf{G} = Sp(2n) = \{g \in SL(2n) | \, {}^t gJg = J\}$. Then $\mathfrak{g} = \{X \in M(n) | \, {}^t XJ + JX = 0\}$.

**Definition 1.12.** *We have an adjoint representation* $Ad : \mathbf{G} \longrightarrow End(\mathfrak{g})$*, defined by* $Ad(g)(X) = gXg^{-1}$.

Let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$. Then $Ad(\mathbf{T})$ is a set of diagonalizable commuting endomorphisms $\mathfrak{g} \longrightarrow \mathfrak{g}$. Hence they are simultaneously diagonalizable. Eigenvalues are characters of $\mathbf{T}$. Hence we have

$$\mathfrak{g} = \mathfrak{g}_0^{(\mathbf{T})} \oplus \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha^{(\mathbf{T})},$$

where $\mathfrak{g}_\alpha^{(\mathbf{T})} = \{X \in \mathfrak{g} | \, Ad(t)(X) = \alpha(t)X\}$, and $\alpha \in X^*(\mathbf{T})$, $\alpha \neq 0$. Only finitely many such $\alpha$'s appear. We call $\Phi$ the set of roots of $\mathbf{G}$ with respect to $\mathbf{T}$.

*Example 1.13.* Let $\mathbf{G} = Sp(4)$. Then $\mathbf{T} = \{x(t_1, t_2) = diag(t_1, t_2, t_2^{-1}, t_1^{-1})\}$ and $\mathfrak{g} = \{\begin{pmatrix} A & B \\ C & A' \end{pmatrix}\}$, where $A = \begin{pmatrix} u & v \\ w & x \end{pmatrix}, A' = \begin{pmatrix} -x & -v \\ -w & -u \end{pmatrix}$, and $B, C$ are of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. The roots are $\{\pm(e_1 \pm e_2), \pm 2e_1, \pm 2e_2\}$, where $e_1 : x(t_1, t_2) \mapsto t_1, e_2 : x(t_1, t_2) \mapsto t_2$. Then $\mathfrak{g}_{e_1 - e_2} = \{\begin{pmatrix} A & \\ & A' \end{pmatrix}\}$, where $A = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$, $\mathfrak{g}_{2e_1} = \{\begin{pmatrix} O & B \\ O & O \end{pmatrix}\}$, where $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, $\mathfrak{g}_{2e_2} = \{\begin{pmatrix} O & B \\ O & O \end{pmatrix}\}$, where $B = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$, $\mathfrak{g}_{e_1 + e_2} = \{\begin{pmatrix} O & B \\ O & O \end{pmatrix}\}$, where $B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Let $N(\mathbf{T}), Z(\mathbf{T})$ be the normalizer and centralizer of $\mathbf{T}$ in $\mathbf{G}$. Then $Z(\mathbf{T}) = \mathbf{T}$ and $W = N(\mathbf{T})/\mathbf{T}$ is finite, called the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}$. For $s \in N(\mathbf{T})$, we can define $w_s : \mathbf{T} \longrightarrow \mathbf{T}$, by $w_s(t) = sts^{-1}$. It induces an isomorphism $w_s' : X^*(\mathbf{T}) \longrightarrow X^*(\mathbf{T})$ by $w_s'(\chi) = \chi \circ w_s$. We will identify $s$ with $w_s$ and $w_s'$.

Let $X_*(\mathbf{T}) = Hom(GL(1), \mathbf{T})$ be the group of cocharacters. Then there is a natural pairing $<, >$: $X^*(\mathbf{T}) \times X_*(\mathbf{T}) \longrightarrow \mathbb{Z}$. For $\chi \in X^*(\mathbf{T}), \mu \in X_*(\mathbf{T})$, we define $< \chi, \mu > \in \mathbb{Z}$ as follows: $\chi \circ \mu : GL(1) \longrightarrow GL(1)$. Since $X^*(GL(1)) \simeq \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $\chi \circ \mu(t) = t^k$. Define $< \chi, \mu >= k$. Using this pairing, we identify $X_*(\mathbf{T})$ with $Hom(X^*(\mathbf{T}), \mathbb{Z})$.

For each $\alpha \in \Phi$, we define the coroot $\alpha^\vee \in X_*(\mathbf{T})$ as follows: Since $\alpha : \mathbf{T} \longrightarrow GL(1)$, $(ker\alpha)^0 \subset \mathbf{T}$ is a subtorus of codimension one. ($(ker\alpha)^0$ is the connected component of the identity of $ker\alpha$.) Let $Z_\alpha$ be the centralizer of $(ker\alpha)^0$ in $\mathbf{G}$. It is a connected, reductive group with $\mathbf{T}$ as a maximal torus. Let $G_\alpha$ be the derived group of $Z_\alpha$. Then $G_\alpha \simeq SL(2)$ or $PGL(2)$, and $G_\alpha$ has a maximal torus $T_\alpha \subset \mathbf{T}$. Define $\alpha^\vee : GL(1) \longrightarrow T_\alpha$ be the unique homomorphism such that $< \alpha, \alpha^\vee >= 2$.

**Definition 1.14.** *Suppose* $\mathbf{G}$ *is reductive and* $\mathbf{T}$ *is a maximal torus. Then* $(X^*(\mathbf{T}), \Phi, X_*(\mathbf{T}), \Phi^\vee)$ *is called a root datum of* $\mathbf{G}$.

*Examples 1.15.* (1) $\mathbf{G} = GL(n)$. Let $\alpha$ be a root such that $\alpha(diag(t_1, ..., t_n)) = t_i t_{i+1}^{-1}$. Then $ker\alpha = \{diag(t_1, .., t_{i-1}, tI_2, t_{i+2}, ..., t_n)\}$. It is connected. So $Z_\alpha = \{diag(t_1, ..., t_{i-1}, GL(2), t_{i+2}, ..., t_n)\}$ and $G_\alpha = \{diag(1, ..., 1, SL(2), 1, ..., 1)\}$. Hence $\alpha^\vee(t) = (1, ..., 1, t, t^{-1}, 1, ..., 1)$.

(2) $\mathbf{G} = Sp(4)$. Let $\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2$ in Example 1.13. Then $\alpha_1^\vee(t) = diag(t, t^{-1}, t, t^{-1})$ and $\alpha_2^\vee(t) = diag(1, t, t^{-1}, 1)$.

Suppose $\mathbf{G}$ is semi-simple, and $\mathbf{T}$ is a maximal torus. Let $X = X^*(\mathbf{T})$. Then $(X, \Phi, W)$ is a root system: $X$ is a free module of rank $l$ ($l = dim\mathbf{T}$); $\Phi$ is finite subset of $X$; $W$ is a finite automorphism group of $X$ such that

1. $0 \notin \Phi$; if $\alpha \in \Phi$, then $-\alpha \in \Phi$
2. If $\alpha \in \Phi$, and $c\alpha \in \Phi$ for $c \in \mathbb{Q}$, then $c = \pm 1$
3. To each $\alpha \in \Phi$, there corresponds $w_\alpha \in W$ such that $w_\alpha(\chi) = \chi - \alpha^\vee(\chi)\alpha$ for $\chi \in X$. Also $w_\alpha(\Phi) = \Phi$.
4. $X_\mathbb{Q} (= X \otimes_\mathbb{Z} \mathbb{Q})$ is generated by $\Phi$ as a vector space over $\mathbb{Q}$
5. $W$ is generated by $\{w_\alpha : \alpha \in \Phi\}$

If $\mathbf{G}$ is reductive, then we need to take $X = X^*(T/S)$, where $S = Z(\mathbf{G})^0$. Since $W$ is finite, one can introduce $W$-invariant positive definite symmetric bilinear form $( , )$ on $X_\mathbb{Q}$. (Take any positive definite symmetric bilinear form $< , >$ and define $(x, y) = \frac{1}{|W|} \sum_{w \in W} < wx, wy >$.)

¿From the relation $(w_\alpha(\chi), w_\alpha(\chi)) = (\chi, \chi)$ and $w_\alpha(\chi) = \chi - \alpha^\vee(\chi)\alpha$, we have $\alpha^\vee(\chi) = \frac{2(\alpha,\chi)}{(\alpha,\alpha)}$. Using this, we can identify $\alpha^\vee$ with $\frac{2\alpha}{(\alpha,\alpha)}$. Also we see that $w_\alpha^2 = 1$, $w_\alpha(\alpha) = -\alpha$ and $w_\alpha$ leaves fixed the hyperplane $H_\alpha = \{\chi \in X_\mathbb{R} : (\alpha, \chi) = 0\}$. We call $w_\alpha$ "reflection" or "symmetry" with respect to $\alpha$.

For $\alpha, \beta \in \Phi$, $c_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$. It is called Cartan integer.

*1.3 Classification of root systems.* The space $X_\mathbb{R} - \bigcup_{\alpha \in \Phi} H_\alpha$ is a finite union of disjoint connected components; such a component is called a Weyl chamber. To each Weyl chamber $C^0$, is associated a linear order in $X$; $\alpha > 0$ if $(\alpha, \chi) > 0$ for all $\chi \in C^0$. We denote the set of positive roots by $\Phi_+$.

**Definition 1.16.** *A positive root* $\alpha$ *is said to be simple if* $\alpha$ *cannot be expressed in the form* $\beta + \gamma$ *for* $\beta, \gamma \in \Phi_+$.

We denote the set of simple roots of $\Phi_+$ by $\Delta$. It is called a fundamental system.

**Theorem 1.17 (Main properties of fundamental system).**

(1) *The fundamental system consists of $l$ linear independent roots $\alpha_1, ..., \alpha_l$.*

(2) *Every root $\alpha \in \Phi$ can be written uniquely in the form*

$$\alpha = \pm \sum_{i=1}^{l} m_i \alpha_i,$$

*where $m_i \in \mathbb{Z}_+ \cup \{0\}$.*

(3) *$W$ is generated by $\{w_{\alpha_i} : \alpha_i \in \Delta\}$*

(4) *Every root $\alpha \in \Phi$ can be written in the form*

$$\alpha = w_{\alpha_{i_r}} \cdots w_{\alpha_{i_1}} \alpha_{i_0},$$

*where $\alpha_{i_0}, ..., \alpha_{i_r} \in \Delta$*

(5) *$W$ acts simply transitively on the set of Weyl chambers. Namely, there is a one to one correspondence between fundamental systems of $\Phi$ and Weyl chambers.*

**Lemma 1.18.** *If $\alpha, \beta$ are two nonproportional roots, and $(\alpha, \beta) > 0$, then $\alpha - \beta$ is a root.*

*Proof.* Recall that $c_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$. By Cauchy-Schwartz inequality, $c_{\alpha\beta}c_{\beta\alpha} < 4$. Hence if $(\alpha, \beta) > 0$, then $c_{\alpha\beta}$ or $c_{\beta\alpha} = 1$. If $c_{\beta\alpha} = 1$, $\alpha - \beta = \alpha - c_{\beta\alpha}\beta = w_\beta(\alpha) \in \Phi$. The other case is the same. □

**Corollary 1.19.** *For $\alpha_i, \alpha_j \in \Delta$, $(\alpha_i, \alpha_j) \leq 0$, and $c_{\alpha_i,\alpha_j} \in \{0, -1, -2, -3\}$.*

**Definition 1.20.** *A root system $\Phi$ is reducible if $\Phi = \Phi_1 \cup \Phi_2$, where $\Phi_1, \Phi_2$ are non-empty subsystems of $\Phi$ and $\Phi_1 \perp \Phi_2$. A root system $\Phi$ is called irreducible if it is not reducible.*

Given an irreducible fundamental root system $\Delta = \{\alpha_1, .., \alpha_l\}$, we call the matrix $(c_{\alpha_i,\alpha_j})$ Cartan matrix. We can attach Dynkin diagram to $\Delta$ in the following way: To each vector $\alpha_i \in \Delta$, associate a vertex, and connect vertices corresponding to $\alpha_i$ and $\alpha_j$ with a single, double, or triple line according to whether $c_{\alpha_i,\alpha_j} = -1, -2, -3$. The arrows point from a longer to a shorter vector, when the lengths are different.

The following is a list of the Dynkin diagrams of irreducible fundamental systems (See [Se] or [Hum]):

**Theorem 1.21 (Classification of irreducible root systems).**

$A_l\,(SL(l+1))$ : $o$ — $o$ — $\cdots$ — $o$ — $o$

$B_l\,(SO(2l+1))$ : $o$ — $o$ — $\cdots$ — $o$ — $o \Longrightarrow o$

$C_l\,(Sp(2l))$ : $o$ — $o$ — $\cdots$ — $o$ — $o \Longleftarrow o$

$D_l$:

$$\begin{array}{ccccccccccc} o & — & o & — & \cdots & — & o & — & o & — & o \\ & & & & & & & & | & & \\ & & & & & & & & o & & \end{array}$$

$E_6$:

$$
\begin{array}{ccccccccc}
o_1 & — & o_2 & — & o_3 & — & o_4 & — & o_5 \\
 & & & & | & & & & \\
 & & & & o_6 & & & &
\end{array}
$$

$E_7$:

$$
\begin{array}{ccccccccccc}
o_1 & — & o_2 & — & o_3 & — & o_4 & — & o_5 & — & o_6 \\
 & & & & & & | & & & & \\
 & & & & & & o_7 & & & &
\end{array}
$$

$E_8$:

$$
\begin{array}{ccccccccccccc}
o_1 & — & o_2 & — & o_3 & — & o_4 & — & o_5 & — & o_6 & — & o_7 \\
 & & & & & & & & | & & & & \\
 & & & & & & & & o_8 & & & &
\end{array}
$$

$F_4 : o_1 — o_2 \Longleftarrow o_3 — o_4$
$G_2 : o_1 \Lleftarrow o_2$

**Theorem 1.22.** *There exists a one to one correspondence between irreducible root systems and simple Lie algebras.*

Complex Lie groups attached to the above irreducible root systems have been known before Chevalley. Chevalley observed that they can be constructed as algebraic groups. The following theorem gives a one to one correspondence between irreducible root systems and isogeny classes of split simple algebraic groups defined over a prime field (Assume that it is of characteristic zero).

**Theorem 1.23 (Fundamental Theorem of Chevalley).** *(1) Given a root system $(X, \Phi)$, there exists a connected semi-simple algebraic group $\mathbf{G}$, defined over a prime field having $(X, \Phi)$ as its root system (with respect to a split maximal torus $\mathbf{T}$ of $\mathbf{G}$). We denote $\mathbf{G}$ by $G(X, \Phi)$.*

*(2) Suppose $(X_1, \Phi), (X_2, \Phi)$ are two root systems with associated algebraic groups $\mathbf{G}_1, \mathbf{G}_2$. Suppose there is an injection $\rho : X_2 \longrightarrow X_1$ such that $\rho$ is an identity on $\Phi$. Then there exists an isogeny $\phi : \mathbf{G}_1 \longrightarrow \mathbf{G}_2$.*

The above group $G(X, \Phi)$ is called "Chevalley group," or "split group", since it has a maximal torus which is split over the prime field. These days, any split reductive groups are called Chevalley groups.

Let $(X, \Phi)$ be a root system. Let

$$
\begin{aligned}
X_0 &= \{\Delta\}_{\mathbb{Z}} = \{\Phi\}_{\mathbb{Z}}, \\
X^0 &= \{\Phi^\vee\}_{\mathbb{Z}}^{\wedge} = \{\chi \in X_{\mathbb{Q}} : (\chi, \alpha^\vee) \in \mathbb{Z}, \text{for all } \alpha^\vee \in \Phi^\vee\}.
\end{aligned}
$$

$X_0$ is called the root module of $\Phi$; $X^0$ is called the weight module of $\Phi$. We have inclusions: $X_0 \subset X \subset X^0$. By the fundamental theorem of Chevalley, there exist isogenies

$$G(X^0, \Phi) \longrightarrow G(X, \Phi) \longrightarrow G(X_0, \Phi).$$

**Proposition 1.24.** *The center of $G(X^0, \Phi)$ is finite and it is isomorphic to $X^0/X_0$.*

Hence given a root system, there exist only a finitely many Chevalley groups in the isogeny class.

**Definition 1.25.** *The group $G(X^0, \Phi)$ is called simply connected group of type $\Phi$. The group $G(X_0, \Phi)$ is called adjoint group of type $\Phi$. (Note that the center of $G(X_0, \Phi)$ is trivial.)*

*1.4 Construction of Chevalley groups: simply connected type.* Let $\Phi$ be a root system and let $\mathfrak{g}$ be a semi-simple Lie algebra determined by $\Phi$. Hence

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $dim\, \mathfrak{g}_\alpha = 1$ for each $\alpha \in \Phi$. Let $\Delta = \{\alpha_1, ..., \alpha_l\}$ be simple roots; $l = dim\, \mathfrak{h}$. For each $\alpha$, let $H'_\alpha \in \mathfrak{h}$ such that $(H, H'_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}$. Define $H_\alpha = \frac{2}{(\alpha,\alpha)} H'_\alpha$, and write $H_i$ for $H_{\alpha_i}$.

**Theorem 1.26 (Existence of Chevalley basis).** *Given the $H_i$, $i = 1, ..., l$, chosen above, one can find $E_\alpha \in \mathfrak{g}_\alpha$, $E_\alpha \neq 0$, for each $\alpha \in \Phi$ such that $H_i, E_\alpha$ together form a basis for $\mathfrak{g}$ relative to which the equations of structure are as follows:*

(1) $[H_i, H_j] = 0$
(2) $[H_i, E_\alpha] = c_{\alpha_i,\alpha} E_\alpha$
(3) $[E_\alpha, E_{-\alpha}] = H_\alpha =$*integral combination of $H_i$'s*
(4) $[E_\alpha, E_\beta] = \pm(r+1)E_{\alpha+\beta}$ *if $\alpha + \beta$ is a root, where $r$ is such that $\beta - r\alpha$ is a root, and $\beta - (r+1)\alpha$ is not a root.*
(5) $[E_\alpha, E_\beta] = 0$ *if $\alpha + \beta \neq 0$ and $\alpha + \beta \neq 0$ is not a root.*

When $r = 0$, all roots in $\Phi$ have the same lengths and $\Phi$ is called simply laced root system. They are root systems of type $A-D-E$. In this case, we can determine the sign in (4) easily: Let $[E_\alpha, E_\beta] = s_{\alpha\beta} E_{\alpha+\beta}$ if $\alpha+\beta \in \Phi$. Let $A = (c_{\alpha_i,\alpha_j}) = B + {}^tB$ be the Cartan matrix. Here $B$ is an upper triangular matrix. It gives rise to an integral valued bilinear form $B(\alpha, \beta)$ such that $(\alpha, \beta) = B(\alpha) + B(\beta, \alpha)$ and

$$B(\alpha_i, \alpha_j) = \begin{cases} \frac{1}{2}(\alpha_i, \alpha_j), & \text{if } i = j \\ 0, & \text{if } i > j \\ (\alpha_i, \alpha_j), & \text{if } i < j \end{cases}$$

Then $s_{\alpha\beta} = (-1)^{B(\alpha,\beta)}$.

Since $E_\alpha$ is nilpotent, the exponential map $exp(E_\alpha)$ is well-defined. For $t \in \bar{F}$, let $e_\alpha(t) = exp(tE_\alpha)$ for each $\alpha \in \Phi$. Let $U_\alpha = \{e_\alpha(t) : t \in \bar{F}\}$.

**Theorem 1.27.** *The simply connected Chevalley group $\mathbf{G} = G(X^0, \Phi)$ is generated by $U_\alpha$ for all $\alpha \in \Phi$.*

For $t \in \bar{F}^*$, let $w_\alpha(t) = e_\alpha(t)e_{-\alpha}(t^{-1})e_\alpha(t)$ and $h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$. Then $\alpha^\vee(t) = h_\alpha(t)$.

Let $\mathbf{U}$ be the subgroup of $\mathbf{G}$ generated by $U_\alpha$ for all $\alpha \in \Phi_+$, and let $\mathbf{T}$ be the subgroup of $\mathbf{G}$ generated by all $h_\alpha(t)$, $\alpha \in \Phi$. Let $\mathbf{B}$ be the group generated by $\mathbf{U}$ and $\mathbf{T}$. Then $\mathbf{B} = \mathbf{T} \cdot \mathbf{U}$ (semi-direct product) and $\mathbf{T} \cap \mathbf{U} = \{e\}$.

**Theorem 1.28.** **B** *is a Borel subgroup (maximal connected solvable subgroup of* **G***) and* **T** *is a maximal torus.*

*Example 1.29.* Suppose $\mathfrak{g} = sl(2)$. Then $\Phi = \{\alpha, -\alpha\}$ and

$$H_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence

$$h_\alpha(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad e_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad w_\alpha(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}.$$

We will use the following proposition to reduce many calculations on **G** to those of $SL(2)$.

**Proposition 1.30.** *If $\alpha \in \Phi$, there exists an injective homomorphism $\phi_\alpha : SL(2) \longrightarrow$* **G** *such that*

$$\phi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = h_\alpha(t), \quad \phi_\alpha \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e_\alpha(t), \quad \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = w_\alpha(1).$$

Let $F$ be a $p$-adic field and $\mathcal{O}$ be its ring of integers. Let $K$ be the subgroup of $\mathbf{G}(F)$ generated by $\{e_\alpha(t) : t \in \mathcal{O}, \alpha \in \Phi\}$. Then $K$ is a maximal compact subgroup of $\mathbf{G}(F)$, and $\mathbf{G}(F) = K\mathbf{B}(F)$ (Iwasawa decomposition). We usually denote $K$ by $\mathbf{G}(\mathcal{O})$.

Note that any $t \in \mathbf{T}$ can be written uniquely $t = \prod_{i=1}^{l} h_{\alpha_i}(t_i)$, $t_i \in \bar{F}^*$. So the center of **G** is given by

$$Z(G) = \{\prod_{i=1}^{l} h_{\alpha_i}(t_i) : \prod_{i=1}^{l} t_i^{(\beta_j, \alpha_i)} = 1, \quad \text{for all } \beta_j \in \Delta\}.$$

*Example 1.31.* Let $\mathbf{G} = Spin(2n)$ be the simply connected group of type $D_n$. The simple roots are $\Delta = \{\alpha_1 = e_1 - e_2, ..., \alpha_{n-3} = e_{n-3} - e_{n-2}, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$. Then

$$Z(G) = \begin{cases} \{\prod_{i=1}^{n-2} h_{\alpha_i}((-1)^i) h_{\alpha_{n-1}}(-t) h_{\alpha_n}(t), \text{ and } h_{\alpha_{n-1}}(t) h_{\alpha_n}(t) : t^2 = 1\}, & \text{if } n \text{ is even} \\ \{h_{\alpha_1}(t^2) \cdots h_{\alpha_{n-2}}(t^{2(n-2)}) h_{\alpha_{n-1}}(t) h_{\alpha_n}(t^3) : t^4 = 1\}, & \text{if } n \text{ is odd.} \end{cases}$$

We set $c = h_{\alpha_{n-1}}(-1) h_{\alpha_n}(-1)$, and

$$z = \begin{cases} \prod_{i=1}^{n-2} h_{\alpha_i}((-1)^i) h_{\alpha_{n-1}}(-1), & \text{if } n \text{ is even} \\ \prod_{i=1}^{n-2} h_{\alpha_i}((-1)^i) h_{\alpha_{n-1}}(\sqrt{-1}) h_{\alpha_n}(\sqrt{-1}), & \text{if } n \text{ is odd.} \end{cases}$$

Note that $c = z^2$ if $n$ is odd. Hence $Z(G) \simeq \mathbb{Z}/4\mathbb{Z}$ if $n$ is odd, and $Z(G) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if $n$ is even. This fact implies that when $n$ is odd, there is, up to isomorphism, a unique non simply-connected, non-adjoint group of type $D_n$, namely, $SO(2n)$. However, when $n$ is even, there are two non-isomorphic, non simply-connected, non-adjoint group of type $D_n$; one is $SO(2n) \simeq Spin(2n)/\{1, c\}$. The other is $HS(2n) \simeq Spin(2n)/\{1, z\}$, the so-called half-spin group.

*1.5 Structure of parabolic subgroups.* Let $\mathbf{G} = G(X^0, \Phi)$ be the simply connected Chevalley group, which corresponds to a root system $\Phi$. Let **T** be a maximal torus.

**Theorem 1.32.** *There is a one to one correspondence between Borel subgroups containing* $\mathbf{T}$ *and fundamental systems* $\Delta$ *of* $\Phi$*. The correspondence is* $\mathbf{B} = B_\Delta \longleftrightarrow \Delta \subset \Phi$*;*

$$B_\Delta = \mathbf{T} \cdot \prod_{\alpha \in \Phi_+} U_\alpha,$$

*where* $\Phi_+$ *is the set of positive roots in* $\Phi$ *determined by* $\Delta$*.*

¿From now on we fix a Borel subgroup $\mathbf{B}$, i.e., a fundamental system $\Delta$.

**Definition 1.33.** *A subgroup of* $\mathbf{G}$ *which contains a Borel subgroup is called parabolic subgroup of* $\mathbf{G}$*.*

**Theorem 1.34.** *There is a one to one correspondence between parabolic subgroups* $\mathbf{P}$ *containing* $B_\Delta$ *and subset* $\theta \subset \Delta$*. The correspondence is* $\mathbf{P} = P_\theta \longleftrightarrow \theta \subset \Delta$*;*

$$P_\theta = G(\Sigma_\theta) \cdot T_\theta \cdot U_\theta^+ = M_\theta N_\theta,$$

*where* $M_\theta = G(\Sigma_\theta) \cdot T_\theta$ *is the Levi subgroup of* $P_\theta$*, and* $N_\theta = U_\theta^+ = \prod_{\alpha \in \Phi_+ - \Sigma_\theta^+} U_\alpha$ *is the unipotent radical of* $P_\theta$*, where* $\Sigma_\theta^+ = \{\theta\}_\mathbb{Z} \cap \Phi_+$*. Here* $T_\theta = (\cap_{\alpha \in \theta} ker\, \alpha)^0$*, the subtorus of* $\mathbf{T}$ *annihilated by* $\theta$*, and* $G(\Sigma_\theta)$ *is the subgroup generated by* $U_\alpha, \alpha \in \Sigma_\theta = \{\theta\}_\mathbb{Z} \cap \Phi$

**Lemma 1.35 (additional properties of parabolic subgroups).**

1. *$M_\theta$ is the centralizer of $T_\theta$ in $\mathbf{G}$, i.e., $T_\theta$ is the connected component of the center of $M_\theta$.*
2. *$G(\Sigma_\theta)$ is the derived group of $M_\theta$.*
3. *$T_\theta \cap G(\Sigma(\theta))$ is finite.*
4. *$G(\Sigma_\theta)$ is simply connected.*

Especially, the Borel subgroup $\mathbf{B}$ corresponds to the empty set in $\Delta$. Also note that if $\theta_1 \subset \theta_2 \subset \Delta$, then $P_{\theta_1} \subset P_{\theta_2}$. If $\theta = \Delta - \{\alpha\}$ for $\alpha \in \Delta$, $\mathbf{P} = P_\theta$ is called a maximal parabolic subgroup.

*Examples 1.36.* (1) $\mathbf{G} = Sp(2n)$; $\Delta = \{e_1 - e_2, e_2 - e_3, ..., e_{n-1} - e_n, 2e_n\}$. Let $\theta = \Delta - \{2e_n\}$. Then $\mathbf{T} = \{diag(t_1, ..., t_n, t_n^{-1}, ..., t_1^{-1})\}$ and $T_\theta = \{diag(t, ..., t, t^{-1}, ..., t^{-1})\}$. Hence $P_\theta = M_\theta N_\theta$, $M_\theta = \{diag(A, -J_n {}^t A^{-1} J_n) : A \in GL(n)\}$. This is called Siegel parabolic subgroup.

We need the following three examples for our proof of the functoriality of symmetric cube.

(2) ($E_6 - 1$ case) Let $\mathbf{G}$ be a simply connected group of type $E_6$. Let $\theta = \Delta - \{\alpha_3\}$. Let $P_\theta = \mathbf{MN}$ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = h_{\alpha_1}(t^2) h_{\alpha_2}(t^4) h_{\alpha_3}(t^6) h_{\alpha_4}(t^4) h_{\alpha_5}(t^2) h_{\alpha_6}(t^3).$$

By Lemma 1.35, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D \simeq SL_3 \times SL_3 \times SL_2$. We identify $\mathbf{A}$ with $GL_1$. We fix an identification of $\mathbf{M}_D$

and $SL_3 \times SL_3 \times SL_2$ under which the element $h_{\alpha_1}(t)h_{\alpha_2}(t^2)$ goes to the diagonal element $diag(t,t,t^{-2})$ of $SL_3$, $h_{\alpha_4}(t^2)h_{\alpha_5}(t)$ to $diag(t,t,t^{-2})$ of $SL_3$, and $h_{\alpha_6}(t)$ to $diag(t,t^{-1})$ of $SL_2$. We define a map $\bar{f} : \mathbf{A} \times \mathbf{M}_D \longrightarrow GL_1 \times GL_1 \times GL_1 \times SL_3 \times SL_3 \times SL_2$ by

$$\bar{f} : (a(t), x, y, z) \longmapsto (t^2, t^2, t^3, x, y, z).$$

Now, $\mathbf{M} \simeq (GL_1 \times SL_3 \times SL_3 \times SL_2)/S$, where

$$S = \{(a(t), t^2 I_3, t^2 I_3, t^3 I_2) : t^6 = 1\}.$$

We obtain an injection $f : \mathbf{M} \longrightarrow GL_3 \times GL_3 \times GL_2$ so that

$$f(h_{\alpha_3}(t)) = (diag(1,1,t),\ diag(1,1,t),\ diag(1,t)).$$

Since $f$ is rational, it induces an injection

$$f : \mathbf{M}(\mathbb{A}) \longrightarrow GL_3(\mathbb{A}) \times GL_3(\mathbb{A}) \times GL_2(\mathbb{A}),$$

such that $\mathbf{M}(\mathbb{A})(\mathbb{A}^*)^2$ is co-compact in $GL_3(\mathbb{A}) \times GL_3(\mathbb{A}) \times GL_2(\mathbb{A})$, where $(\mathbb{A}^*)^2$ is embedded as a center of the first two factors.

(3) ($E_7-1$ case) Let $\mathbf{G}$ be a simply connected group of type $E_7$. Let $\theta = \Delta - \{\alpha_4\}$. Let $P_\theta = \mathbf{MN}$. Then $\mathbf{A} = \{a(t) :\ t \in \overline{F}^*\}$, where

$$a(t) = h_{\alpha_1}(t^3)h_{\alpha_2}(t^6)h_{\alpha_3}(t^9)h_{\alpha_4}(t^{12})h_{\alpha_5}(t^8)h_{\alpha_6}(t^4)h_{\alpha_7}(t^6).$$

By Lemma 1.35, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D \simeq SL_2 \times SL_3 \times SL_4$. Now we proceed exactly the same way as in $E_6 - 1$ case; under the identification of $\mathbf{M}_D$ with $SL_2 \times SL_3 \times SL_4$, $\mathbf{M} \simeq (GL_1 \times SL_2 \times SL_3 \times SL_4)/S$, where

$$S = \{(a(t), t^6 I_2, t^4 I_3, t^3 I_4) : t^{12} = 1\}.$$

We also construct an injection $f : \mathbf{M} \longrightarrow GL_2 \times GL_3 \times GL_4$ so that

$$f(h_{\alpha_4}(t)) = (diag(1,t),\ diag(1,1,t),\ diag(1,1,1,t)).$$

(4) ($D_n - 2$ case) Let $\mathbf{G} = Spin(2n)$ be a split spin group and $\theta = \Delta - \{\alpha_{n-2}\}$. Let $P_\theta = \mathbf{MN}$: $\mathbf{A} = \{a(t) :\ t \in \overline{F}^*\}$, where

$$a(t) = \begin{cases} h_{\alpha_1}(t)h_{\alpha_2}(t^2)\cdots h_{\alpha_{n-2}}(t^{n-2})h_{\alpha_{n-1}}(t^{\frac{n-2}{2}})h_{\alpha_n}(t^{\frac{n-2}{2}}), & \text{if } n \text{ even} \\ h_{\alpha_1}(t^2)h_{\alpha_2}(t^4)\cdots h_{\alpha_{n-2}}(t^{2(n-2)})h_{\alpha_{n-1}}(t^{n-2})h_{\alpha_n}(t^{n-2}), & \text{if } n \text{ odd} \end{cases}$$

By Lemma 1.35, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D \simeq SL_{n-2} \times SL_2 \times SL_2$. As in the above, we have, $\mathbf{M} \simeq (GL_1 \times SL_{n-2} \times SL_2 \times SL_2)/S$, where

$$S = \begin{cases} \{(a(t), tI_{n-2}, t^{\frac{n-2}{2}} I_2, t^{\frac{n-2}{2}} I_2) : t^{n-2} = 1\}, & \text{if } n \text{ even} \\ \{(a(t), t^2 I_{n-2}, t^{n-2} I_2, t^{n-2} I_2) : t^{2(n-2)} = 1\}, & \text{if } n \text{ odd} \end{cases}$$

We obtain an injection $f:\mathbf{M}\longrightarrow GL_{n-2}\times GL_2\times GL_2$ so that

$$f(h_{\alpha_{n-2}}(t))=(diag(1,...,1,t),\, diag(1,t),\, diag(1,t)).$$

We need the following example for our proof of the functoriality of symmetric fourth.

(5) ($D_n-3$ case) Let $\mathbf{G}=Spin(2n)$ be a split spin group and $\theta=\Delta-\{\alpha_{n-3}\}$. Let $P_\theta=\mathbf{MN}$: $\mathbf{A}=\{a(t):\ t\in\overline{F}^*\}$, where

$$a(t)=\begin{cases} h_{\alpha_1}(t^2)h_{\alpha_2}(t^4)\cdots h_{\alpha_{n-3}}(t^{2(n-3)})h_{\alpha_{n-2}}(t^{2(n-3)})h_{\alpha_{n-1}}(t^{n-3})h_{\alpha_n}(t^{n-3}), & \text{if } n \text{ even}\\ h_{\alpha_1}(t)h_{\alpha_2}(t^2)\cdots h_{\alpha_{n-3}}(t^{n-3})h_{\alpha_{n-2}}(t^{n-3})h_{\alpha_{n-1}}(t^{\frac{n-3}{2}})h_{\alpha_n}(t^{\frac{n-3}{2}}), & \text{if } n \text{ odd}\end{cases}$$

By Lemma 1.35, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D\simeq SL_{n-3}\times SL_4$. Now, $\mathbf{M}\simeq(GL_1\times SL_{n-3}\times SL_4)/S$, where

$$S=\begin{cases}\{(a(t),t^2I_{n-3},t^{n-3}I_4):t^{2(n-3)}=1\}, & \text{if } n \text{ even}\\ \{(a(t),tI_{n-3},t^{\frac{n-3}{2}}I_4):t^{n-3}=1\}, & \text{if } n \text{ odd}\end{cases}$$

We obtain an injection $f:\mathbf{M}\longrightarrow GL_{n-3}\times GL_4$ so that

$$f(h_{\alpha_{n-3}}(t))=(diag(1,...,1,t),\, diag(1,1,t,t)).$$

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