Set Theory and its Impact on Analysis

Juris Steprāns — *York University*

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- The proof that analytic sets are measurable and a closer analysis of the rest of the projective hierarchy carried out by Suslin, Lusin, Alexandroff and others;
- and, of course, the Hausdorff-Banach-Tarski paradox.

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- However, recently set theoretic techniques have been used to attain absolute results.

1 Planar geometry

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- There is a set of 17 points in the plane that can not be extended to meet $\mathbb{Z}^2$ except by adding the third corner of a right angle triangle with integer legs:

$$\left(\frac{216}{5},\frac{2}{5}\right)\left(\frac{107}{5},\frac{4}{5}\right)\left(\frac{283}{5},\frac{1}{5}\right)\left(\frac{174}{5},\frac{3}{5}\right)\left(\frac{677}{13},\frac{5}{13}\right)\left(\frac{340}{13},\frac{10}{13}\right)$$

$$\left(\frac{744}{13},\frac{2}{13}\right)\left(\frac{407}{13},\frac{7}{13}\right)\left(\frac{70}{13},\frac{12}{13}\right)\left(\frac{474}{13},\frac{4}{13}\right)\left(\frac{137}{13},\frac{9}{13}\right)\left(\frac{541}{13},\frac{1}{13}\right)$$

$$\left(\frac{204}{13},\frac{6}{13}\right)\left(\frac{712}{13},\frac{11}{13}\right)\left(\frac{271}{13},\frac{3}{13}\right)\left(\frac{779}{13},\frac{8}{13}\right)\left(\frac{2601}{65},\frac{57}{65}\right)$$

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- *Exercise: A. Miller and W. Weiss:* There is no Steinhaus set for a square.
- *Open Question:* Is there a Borel set in the plane meeting each line at precisely 2 points?

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- The methods used for these results are not Ramsey theoretic but have their roots in transfinite inductive constructions of Boolean algebras.

3 Classifying finite rank abelian groups

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- If G is a torsion free abelian group, $0 \neq x \in G$ and p is a prime then the p-height of x is the supremum of all n such that $x = yp^n$ for some $y \in G$. Denote this by $\chi_x(p)$.

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- Baer has shown that if G and H are rank 1 torsion free abelian groups then G and H are isomorphic if and only if $\chi(G) = \chi(H)$.

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- The proof uses notions of descriptive set theory and some deep results of Margulis and Zimmer about group actions in Ergodic theory.