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Distribution of modular symbols

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- Modular symbols and their distribution

- Motivation: Conjectures of Szpiro

  and Goldfeld

- Eisenstein series with modular symbols

- Eisenstein series with characters χ_ϵ and Perturbations of the Laplacian

- Open problems

Modular Symbols

$f(z)$ cusp form of weight 2 for Γ e.g. $\Gamma_0(N)$

$f(z)dz$ 1-form on $X = \Gamma \setminus \mathbf{H}$ e.g.

$$X_0(N) = \Gamma_0(N) \setminus \mathbf{H}$$

$$\langle \gamma, f \rangle = -2\pi i \int_{\tau}^{\gamma\tau} f(z)\, dz$$

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$$

Petersson norm:

$$||f||^2 = \int_{\Gamma\backslash\mathbf{H}} y^2 |f(z)|^2 \, dx dy / y^2$$

Theorem 1 The modular symbols normalized as

$$\widetilde{\langle \gamma, f \rangle} = \sqrt{\frac{\mathrm{vol}(\Gamma \setminus \mathbf{H})}{8\pi^2 ||f||^2}} \frac{\langle \gamma, f \rangle}{\sqrt{\log(c^2 + d^2)}}$$

have a binormal Gaussian distribution with correlation coefficient 0.

This means:

Let

$$(\Gamma_\infty \setminus \Gamma)^T = \{\gamma \in \Gamma_\infty \setminus \Gamma, c^2 + d^2 \leq T\}$$

Then

$$\frac{\#\left\{\gamma \in (\Gamma_\infty \setminus \Gamma)^T | \widetilde{\langle \gamma, f \rangle} \in R\right\}}{\#(\Gamma_\infty \setminus \Gamma)^T}$$
$$\to \frac{1}{2\pi} \int_R \exp\left(-\frac{x^2 + y^2}{2}\right) dxdy$$

as $T \to \infty$.

Theorem 1' Let α be a harmonic real-valued 1-form. Let

$$\langle \gamma, \alpha \rangle = -2\pi i \int_{\tau}^{\gamma\tau} \alpha$$

and

$$\widetilde{\langle \gamma, \alpha \rangle} = \sqrt{\frac{\mathrm{vol}(\Gamma \setminus \mathbf{H})}{8\pi^2 ||f||^2}} \frac{\langle \gamma, f \rangle}{i\sqrt{\log(c^2 + d^2)}}$$

have Gaussian distribution i.e.

$$\frac{\# \left\{ \gamma \in (\Gamma_\infty \setminus \Gamma)^T | \widetilde{\langle \gamma, \alpha \rangle} \in [a, b] \right\}}{\#(\Gamma_\infty \setminus \Gamma)^T}$$
$$\to \frac{1}{\sqrt{2\pi}} \int_a^b \exp\left(-\frac{x^2}{2} \right) dx$$

as $T \to \infty$.

Pairing:

$$H_1(X, \mathbf{Z}) \times H^1_{dR}(X, \mathbf{R}) \to \mathbf{R}$$

$$\langle c, \omega \rangle = \int_c \omega$$

Counting prime closed geodesics on X:

$$\pi_\Gamma(x) = \#\{\{\mathcal{P}\} \,|N(\mathcal{P}) \le x\} \sim \mathrm{li}(x),$$

$N(\mathcal{P}) = e^{\mathrm{length}(\mathcal{P})}$.

Counting geodesics in a homology class:

$\phi : \Gamma \to H_1(X, \mathbf{Z})$, $\beta \in H_1(X, \mathbf{Z})$

$\pi_\beta(x) = \#\{\{\mathcal{P}\} \,|N(\mathcal{P}) \le x, \phi(\mathcal{P}) = \beta\} \sim \dfrac{c_0 x}{\ln(x)^{g+1}}$

g genus.

Motivation:

$$H(\gamma) = \langle \gamma, f \rangle = -2\pi i \int_{\tau}^{\gamma\tau} f(z)\, dz$$

$$\langle \gamma_1\gamma_2, f \rangle = \langle \gamma_1, f \rangle + \langle \gamma_2, f \rangle$$

$$H : \Gamma_0(N) \to \Lambda = \{n_1\Omega_1 + n_2\Omega_2, n_i \in \mathbf{Z}\}$$

$E : \Lambda \setminus \mathbf{C}$ elliptic curve $y^2 = 4x^3 - ax - b$

e_i roots of $4x^3 - ax - b$,

$$D = \prod_{i<j} (e_j - e_i)^2 \qquad \text{discriminant}$$

$$\Omega_1 = 2\int_{e_3}^{\infty} \frac{dx}{\sqrt{4x^3 - ax - b}},$$

$$\Omega_2 = 2\int_{e_2}^{e_3} \frac{dx}{\sqrt{4x^3 - ax - b}}$$

$$\begin{array}{c} X_0(N) \\ \downarrow \\ E \end{array}$$

Szpiro's conjecture: $D \ll N^{\kappa}$.

Goldfeld's conjecture:

$$\langle \gamma, f \rangle = n_1(\gamma, f)\Omega_1 + n_2(\gamma, f)\Omega_2$$

$$n_i \ll N^k, \quad \text{if} \quad c \le N^2$$

$L(f, 1)$ in terms of modular symbols

Eisenstein series (non holomorphic)

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s$$

$$\Delta E(z,s) + s(1-s)E(z,s) = 0$$

Fourier series:

$$\begin{aligned} E(z,s) &= y^s \\ &\quad + \sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} y^{1-s} + \cdots \end{aligned}$$

$$\mathrm{Res}_{s=1} = \frac{3}{\pi}$$

$$\sum_{|cz+d|^2 \le X} 1 \sim \pi \frac{X}{y}$$

Eisenstein series with modular symbols

$$E^{1,0}(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \langle\gamma, f\rangle \Im(\gamma z)^s$$

Goldfeld, O'Sullivan 1998

Chinta, Diamantis

$$E^{1,0}(\gamma z,s) = E^{1,0}(z,s) - \langle\gamma, f\rangle E(z,s)$$

$$\Delta E^{1,0}(z,s) + s(1-s)E^{1,0}(z,s) = 0$$

Residue at $s = 1$ is

$$R(z) = \frac{2\pi i}{\mathrm{vol}(\Gamma \setminus \mathbf{H})} \int_{i\infty}^{z} f.$$

Goldfeld conjectured:

$$\sum_{c^2+d^2\le X, N|c} \langle\gamma, f\rangle = R(i)X + O(X^{1-\epsilon})$$

Application:

$$\Gamma_0(11) = \langle A, B, P_0, P_\infty, ABA^{-1}B^{-1}P_0P_\infty = 1\rangle$$

where

$$A = \begin{pmatrix} -7 & -1 \\ 22 & 3 \end{pmatrix}, B = \begin{pmatrix} 4 & 1 \\ -33 & -8 \end{pmatrix},$$

$$P_0 = \begin{pmatrix} 1 & 0 \\ -11 & 1 \end{pmatrix}, P_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\log_A(A^{e_1}B^{d_1}A^{e_2}B^{d_2}\cdots) = \sum e_i$$

Goldfeld-O'Sullivan:

$$\sum_{|cz+d|^2\le T} \log_A \gamma \;=\; \frac{T}{y}\Re(c\int_\infty^z \eta^2(z)\eta^2(11z)dz) + O(T^{1-\epsilon})$$

Eisenstein series twisted by higher powers of modular symbols:

$$E^{m,n}(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \langle \gamma, f \rangle^m \overline{\langle \gamma, h \rangle}^n \Im(\gamma z)^s$$

Goldfeld conjectured:

Let $f = h$. Then

$E^{1,1}(z,s)$ (involves $|\langle \gamma, f \rangle|^2$) has pole of order 1 at $s = 1$ with residue involving Petersson product $\langle f, f \rangle$

Conjecture:

$$\sum_{c^2+d^2 \le X, N|c} |\langle \gamma, f \rangle|^2 \sim R^*(i)X$$

Theorem 2 $E^{m,n}(z,s)$ converge absolutely in $\Re(s) > 1$ and have meromorphic continuation to all of $\mathbf{C}$.

Corollary 3 $\langle \gamma, f \rangle = O_\epsilon((c^2 + d^2)^\epsilon)$.

Theorem 4 For $1/2 < \Re(s) \leq 1$, $z \in K$

$$E^{m,n}(z,s) \ll_{K,\sigma} t^{(6(m+n)-1)(1-\sigma)+\epsilon}$$

Theorem 5 $E^{2,0}(z,s)$ (involves $\langle \gamma, f \rangle^2$) has simple pole at $s = 1$ with residue

$$\frac{-4\pi^2 (\int_{i\infty}^{z} f(\tau) d\tau)^2}{\mathrm{vol}(\Gamma \setminus \mathbf{H})}$$

Theorem 6 $E^{1,1}(z,s)$ (involves $|\langle\gamma, f\rangle|^2$) has double pole at $s=1$ with singular part

$$\frac{16\pi^2\langle f,f\rangle}{\mathrm{vol}(\Gamma\setminus \mathbf{H})^2(\mathbf{s}-\mathbf{1})^2}$$

$$+\frac{4\pi^2}{\mathrm{vol}(\Gamma\setminus \mathbf{H})(\mathbf{s}-\mathbf{1})}\Big(\Big|\int_{i\infty}^{z} f(\tau)d\tau\Big|^2$$

$$+4\int_{\Gamma\setminus\mathbf{H}}(E_0(z')-r_0(z,z'))y'^2|f(z')|^2d\mu(z')\Big)$$

Theorem 7

$$\sum_{|cz+d|^2\leq X}\langle\gamma, f\rangle^2=\frac{-4\pi^2(\int_{i\infty}^{z} f(\tau)d\tau)^2}{y\mathrm{vol}(\Gamma\setminus \mathbf{H})}X+O(X^{1-\epsilon})$$

$$\sum_{|cz+d|^2\leq X}|\langle\gamma, f\rangle|^2 = \frac{16\pi^2\langle f,f\rangle}{y\mathrm{vol}(\Gamma\setminus \mathbf{H})}X\log X + O(X)$$

Theorem 8 $E^{m,m}(z,s)$ (involves $|\langle\gamma,f\rangle|^{2m}$) has pole of order $m+1$ at $s=1$ with coefficient of $(s-1)^{-m-1}$ equal to

$$\frac{(16\pi^2)^m m!^2 ||f||^{2m}}{\mathrm{vol}(\Gamma\setminus\mathbf{H})^{m+1}}.$$

Theoren 9 $E^{3,0}(z,s)$ (involves $\langle\gamma,f\rangle^3$) has a simple pole at $s=1$ with residue

$$\frac{1}{\mathrm{vol}(\Gamma\setminus\mathbf{H})}\left(2\pi i\int_{i\infty}^{z} f(z)\,dz\right)^3,$$

while $E^{2,1}(z,s)$ (involves $|\langle\gamma,f\rangle|^2\langle\gamma,f\rangle$) has a double pole with leading coefficient

$$\frac{32\pi^2}{\mathrm{vol}(\Gamma\setminus\mathbf{H})^2}\left(2\pi i\int_{i\infty}^{z} f(z)\,dz\right)||f||^2.$$

Eisenstein series with characters χ_ϵ.

$$E_\epsilon(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \chi_\epsilon(\gamma) \Im(\gamma z)^s$$

where

$$\chi_\epsilon(\gamma) = \exp(-2\pi i \epsilon \int_\tau^{\gamma\tau} f)$$

Idea (Petridis, 2002)

$$\frac{d}{d\epsilon} E_\epsilon(z,s)_{\epsilon=0} = E^{1,0}(z,s)$$

$$E_\epsilon(\gamma z, s) = \bar{\chi}_\epsilon(\gamma) E_\epsilon(z,s)$$

Perturbations of the Laplace operator:

$$U_\epsilon : L^2(\Gamma \setminus \mathbf{H}) \to L^2(\Gamma \setminus \mathbf{H}, \bar{\chi}_\epsilon)$$

$$(U_\epsilon h)(z) = \exp(2\pi i \epsilon \int_{i\infty}^{z} w) h(z)$$

$$\Delta : L^2(\Gamma \setminus \mathbf{H}, \bar{\chi}_\epsilon) \to L^2(\Gamma \setminus \mathbf{H}, \bar{\chi}_\epsilon)$$

$$L_\epsilon = U_\epsilon^{-1} \Delta U_\epsilon : L^2(\Gamma \setminus \mathbf{H}) \to L^2(\Gamma \setminus \mathbf{H})$$

History:

P. Sarnak: On cusp forms II, 1989: deformations in character varieties

Phillips-Sarnak condition:

$L(f \otimes \phi_j, s_j + 1/2) \neq 0$

destruction of cusp forms

R. Phillips+P. Sarnak 1992:

$\Re({s_j}^{(2)}) = -|L(f \otimes \phi_j, s_j + 1/2)|^2$

Petridis (2000): perturbation of scattering poles for $\Gamma_0(p)$:

A positive proportion of scattering poles moves off the line $\Re(s) = 1/4$, if $L(f, 1) \neq 0$.

RH$\Leftrightarrow$ scattering poles $\Re(s) = 1/4$.

Theorem 10 (O'Sullivan 1998, Petridis 2002) At a cuspidal eigenvalue $s_j(1-s_j)$ of Δ with cusp forms $\phi_l(z)$, $l = 1, \ldots, N$, $E^{1,0}(z,s)$ has a simple pole with residue

$$\sum_{l=1}^{N} \frac{1}{\pi^{s_j}} L(f \otimes \phi_l, s_j + 1/2)\Gamma(s_j - 1/2)\phi_l(z).$$

Remark 1 If the special values

$$L(f \otimes \phi_l, s_j + 1/2) = 0,$$

then there is no pole at s_j and the value of $E^{1,0}(z,s)$ at s_j is

$$\sum_{l=1}^{N} L'(f \otimes \phi_l, s_j + 1/2)\phi_l(z) + \text{other terms}$$

(up to Gamma factors)

Remark 2 For $\Gamma_0(q)$ double pole of $E^{1,0}(z,s)$ at $s = s_0$, $s_0 = \rho/2$. Assume GRH, simplicity of zeros.

Coefficient of $(s-s_0)^{-2}$:

$$\text{Gamma factors} \cdot L(f,1)L(f,2-2s_0)$$

If $L(f,1)=0$, then residue is

$$\text{junk } \cdot L'(f,1)L(2-2s_0) + \text{other terms}$$

$$E_\epsilon(z,s) = U_\epsilon D(z,s,\epsilon)$$

$$(\Delta + s(1-s))E_\epsilon(z,s) = 0$$

gives

$$(L_\epsilon + s(1-s))D(z,s,\epsilon) = 0$$

Differentiate:

$$\dot{L}D(z,s,0) + (\Delta + s(1-s))\dot{D}(z,s,0) = 0$$

$$\dot{D}(z,s,0) = -R(s)(\dot{L}E(z,s))$$

where

$$R(s) = (\Delta + s(1-s))^{-1}$$

automorphic resolvent of Δ.

Growth on vertical lines

$$\phi(s) = \frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)} ab^{1-2s} L(s)$$

$$L(s) = 1 + \sum_j \frac{a_j}{l_j^s}, \qquad l_j > 1$$

Phragmén-Lindelöf on $E(z,s)/L(s)$

$$\dot{D}(z,s) = -R(s)(\dot{L}E(z,s))$$

$$||R(z)||_\infty \le \frac{1}{\text{dist}(z, \text{Spec A})}$$

Sobolev imbedding.

Questions:

Cocompact (quaternion) groups Γ.

Risager: Hyperbolic Eisenstein series around γ, gets Gauss distribution if

$$\int_\gamma f = 0$$

Eisenstein series twisted by period polynomials of cusp forms of weight $k > 2$.

Higher-rank: $\mathsf{SL}_n(\mathbf{R})$ (Goldfeld-Gunnells).