Bifurcations of the Forced van der Pol Equation

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Forced van der Pol Equation

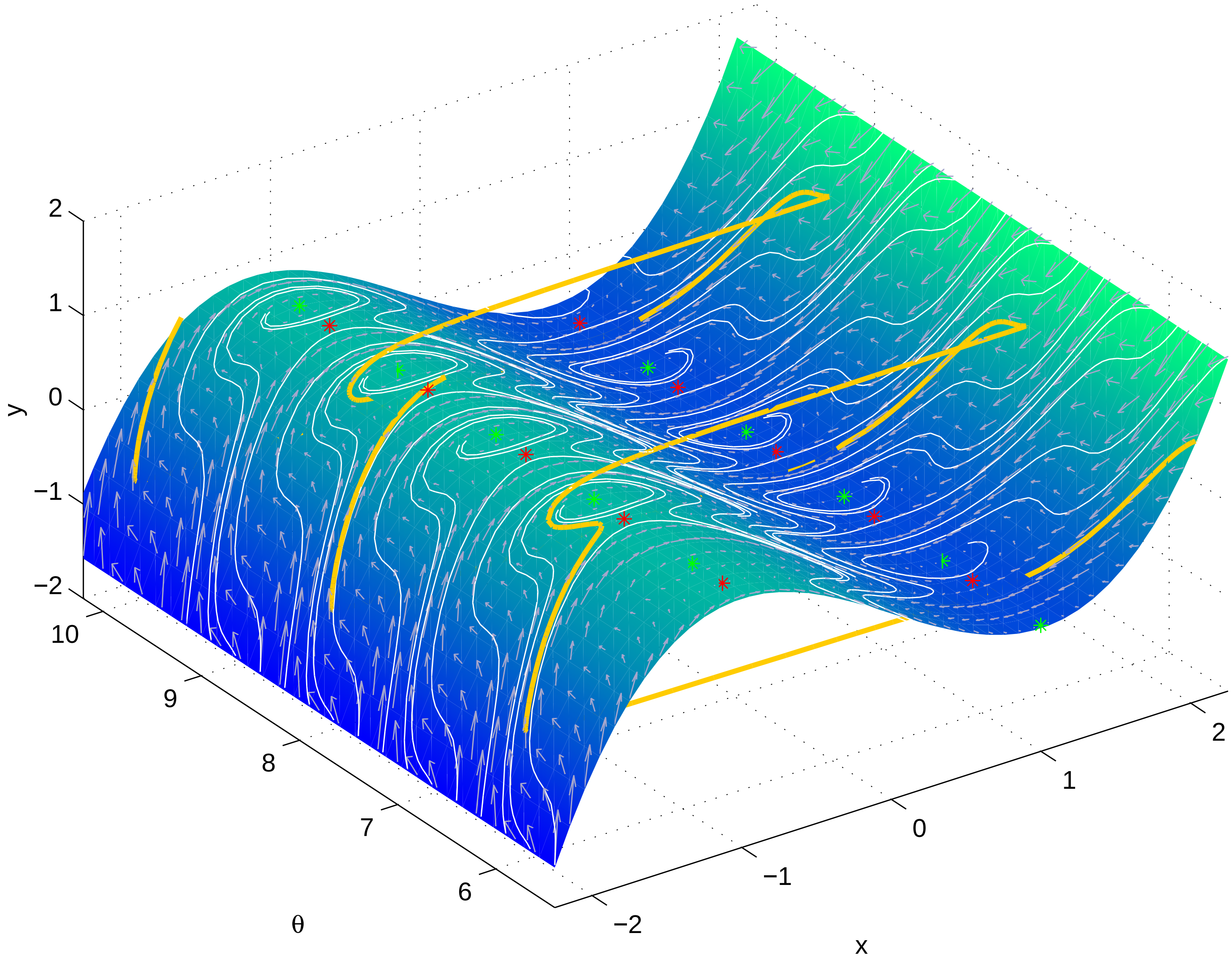
$$
\begin{aligned}
\varepsilon \dot{x} &= y + x - \frac{x^3}{3} \\
\dot{y} &= -x + a\sin(2\pi\theta) \\
\dot{\theta} &= \omega
\end{aligned}
$$

Classical model in dynamical systems theory: first example of "chaos"

- Analysis by Cartwright and Littlewood, Grasman, Takens, ...
- Emphasis upon stable subharmonic orbits of different periods
- Levinson's piecewise linear modification the precursor to Smale's horseshoe

No published calculations for $\varepsilon < 0.01$?

The Forced Van der Pol Equation



Bifurcation with Multiple Time Scales

Dynamical systems theory examines generic (persistent) phenomena

- Special structure (multiple time scales) changes meaning of generic
- Forced van der Pol system provides case study for bifurcations of relaxation oscillations
- Limited understanding of bifurcation in slow-fast systems
- Numerical methods for multiple time scales are problematic

Slow-fast Systems

$$\begin{aligned} \varepsilon \dot{x} &= f(x,y) \qquad x \in R^m \\ \dot{y} &= g(x,y) \qquad y \in R^n \end{aligned}$$

Two-time scales

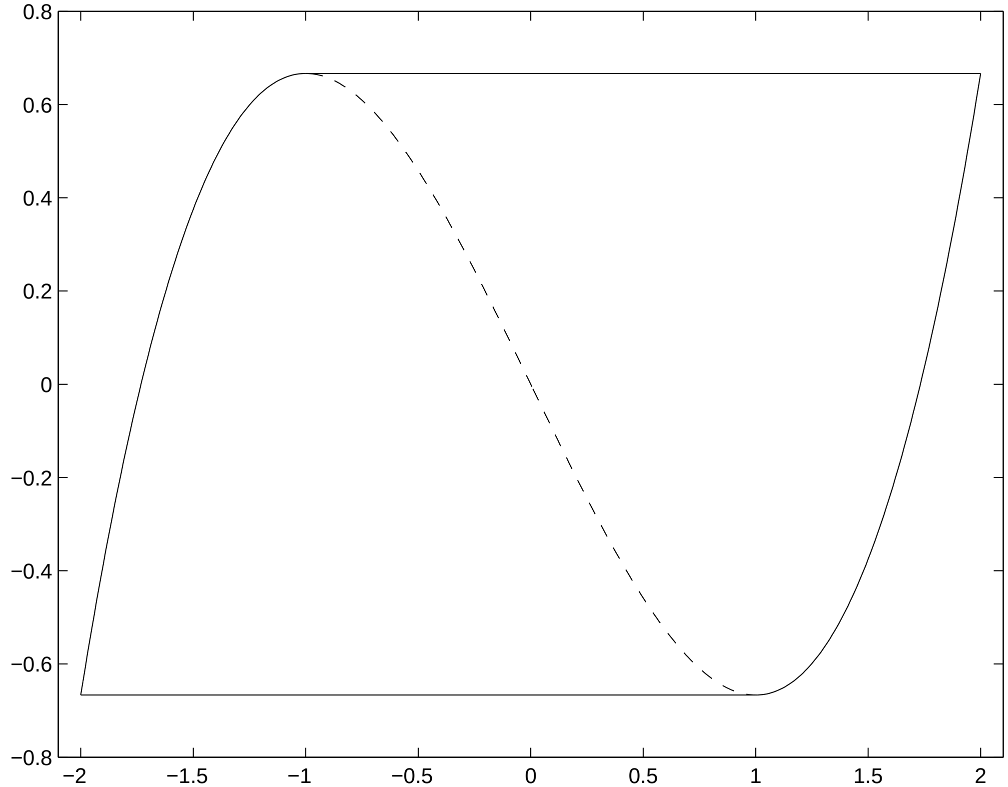
- Limit $\epsilon = 0$ is **differential algebraic equation**

Time rescaling produces **slowly varying** system

$$\begin{aligned} x' &= f(x,y) \qquad x \in R^m \\ y' &= \varepsilon g(x,y) \qquad y \in R^n \end{aligned}$$

- For fixed y, flow in x is **fast subsystem**

Van der Pol Cycle



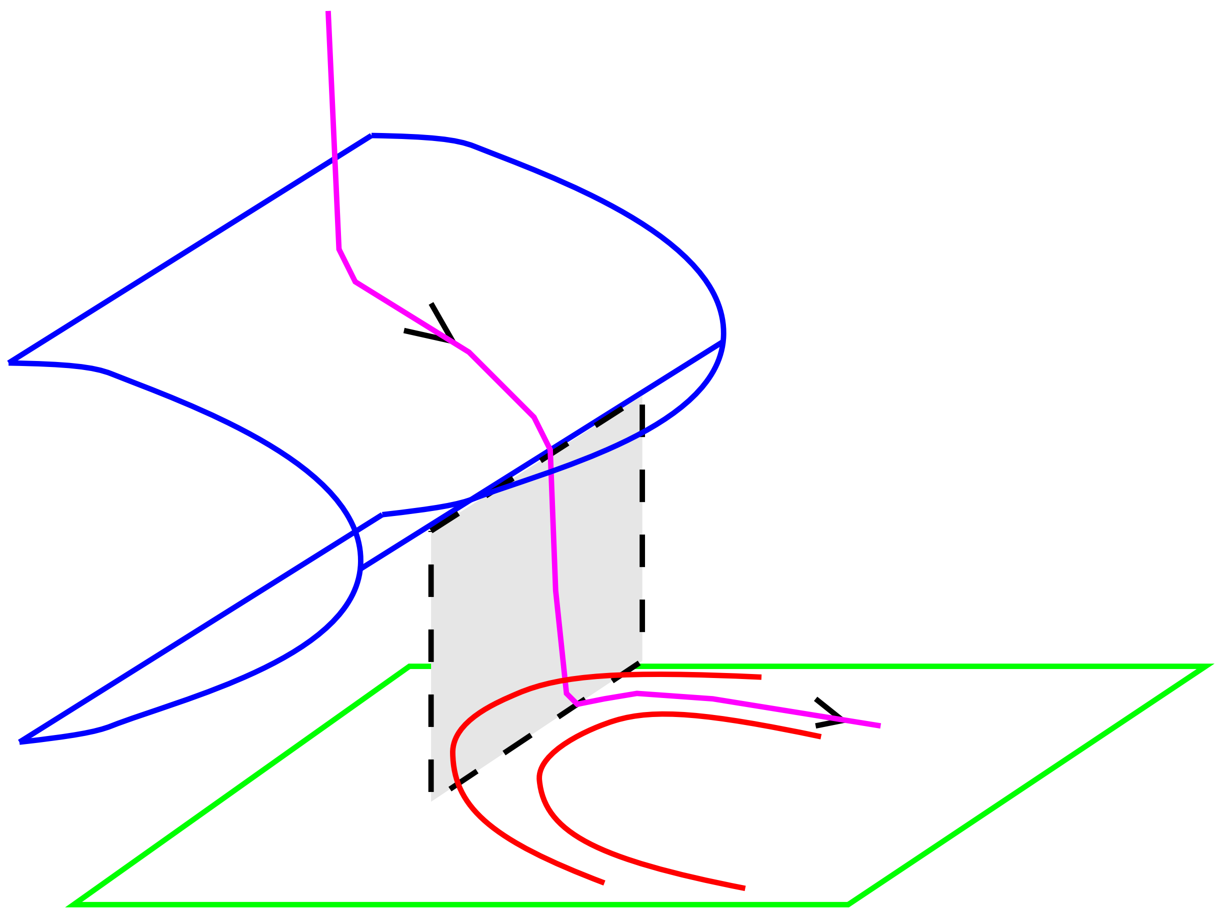
Segments of the cubic characteristic and horizontal segments form limit of periodic orbits as $\epsilon \to 0$

Terminology

Standing assumption: limit sets of fast subsystems are equilibria

- **Critical manifold**: set of equilibria from fast subsystems
- **Slow manifold**: invariant manifold on which flow has speed $O(\varepsilon)$
- **Slow flow**: flow on critical manifold derived by rescaling time and eliminating fast variables
- **Fold**: singularities of the projection of the critical manifold onto the slow variables
- **Junctions**: where slow and fast segments of a trajectory meet
- **Relaxation oscillation**: periodic orbit with slow and fast segments

Slow-fast Flow



Forced van der Pol Slow Flow

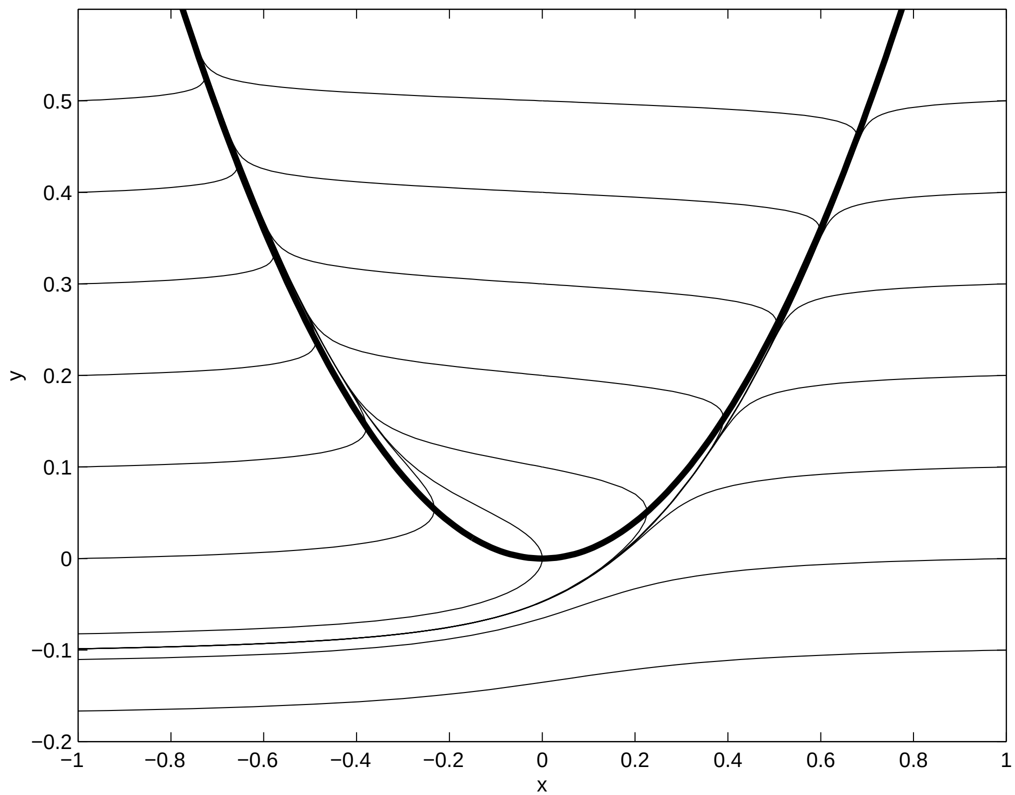
On the critical manifold

$$\dot{y} - (x^2 - 1)\dot{x} \quad = \quad 0$$

- Rescale by $h(x, y, \theta) = x^2 - 1$
- Eliminate y from rescaled slow equations

$$\begin{aligned} \theta' &= \omega(x^2 - 1) \\ x' &= -x + a\sin(2\pi\theta) \end{aligned}$$

- Jumps from fold curves $x = \pm 1$ to $x = \mp 2$
- Symmetry: $x \to -x$, $\theta \to \theta + 0.5$

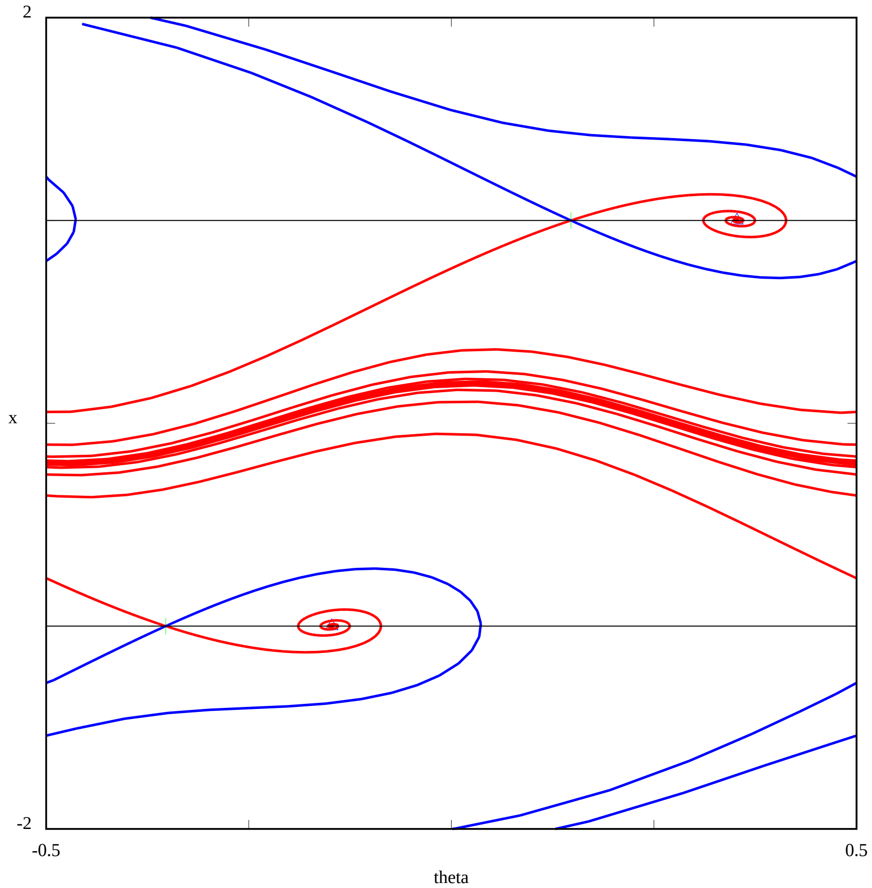


Flow past generic fold

The Singular Limit: Classical Theory

Relate slow flow and fast trajectories to trajectories of full flow?

- Theorem: On regular sheets of critical manifold, slow flow trajectories are singular limits of trajectories of full system
- Theorem: Flow along jumps of regular folds bounding stable slow manifolds are limits of trajectories of full system
- **Folded singularities**: singular points of slow flow on fold curves of critical manifold
- **Canards**: trajectories that flow along unstable sheets of critical manifold form folded singularities



Forced van der Pol slow flow: $a = 1.5\ \omega = 1$

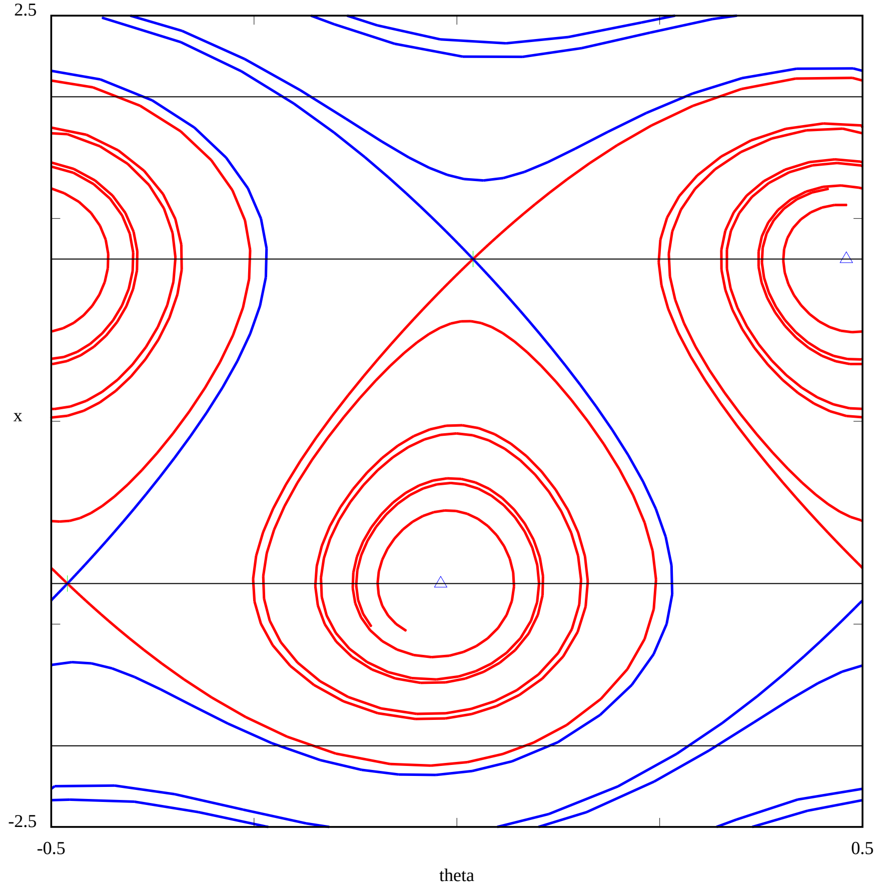
Slow Flow Geometry

Folded singularities if $a > 1$: $x = \pm 1$, $\sin(2\pi\theta) = \pm 1/a$

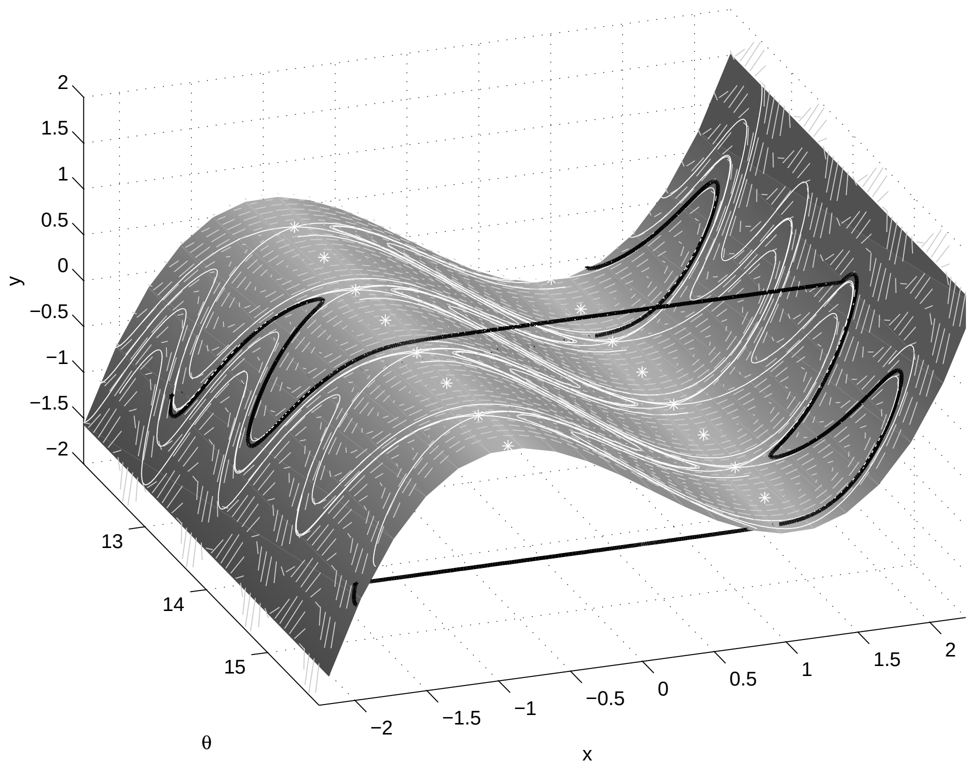
- Folded saddles and nodes ($1 < a < \sqrt{1 + 1/(256\pi^2\omega^2)}$) or foci ($a > \sqrt{1 + 1/(256\pi^2\omega^2)}$)
- Stable and unstable manifolds of folded saddles yield discontinuities of return map
- Asymptotics of return map differ on two sides of discontinuities: zero and infinite slopes

Tin (tangency inflow) points if $a > 2$: $x = \pm 2$, $\sin(2\pi\theta) = \pm 2/a$

- Tin points become quadratic turning points in return maps



Forced van der Pol slow flow: $a = 8 \; \omega = 1$

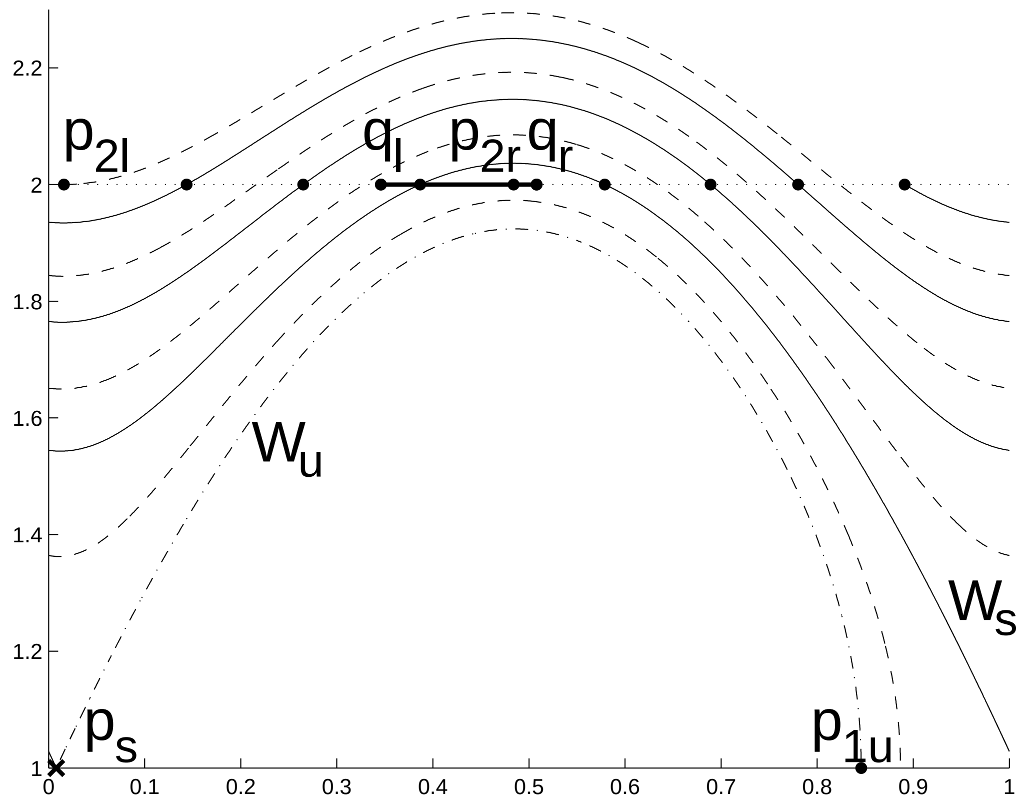


Forced van der Pol 3d flow: $a = 20 \; \omega = 5$

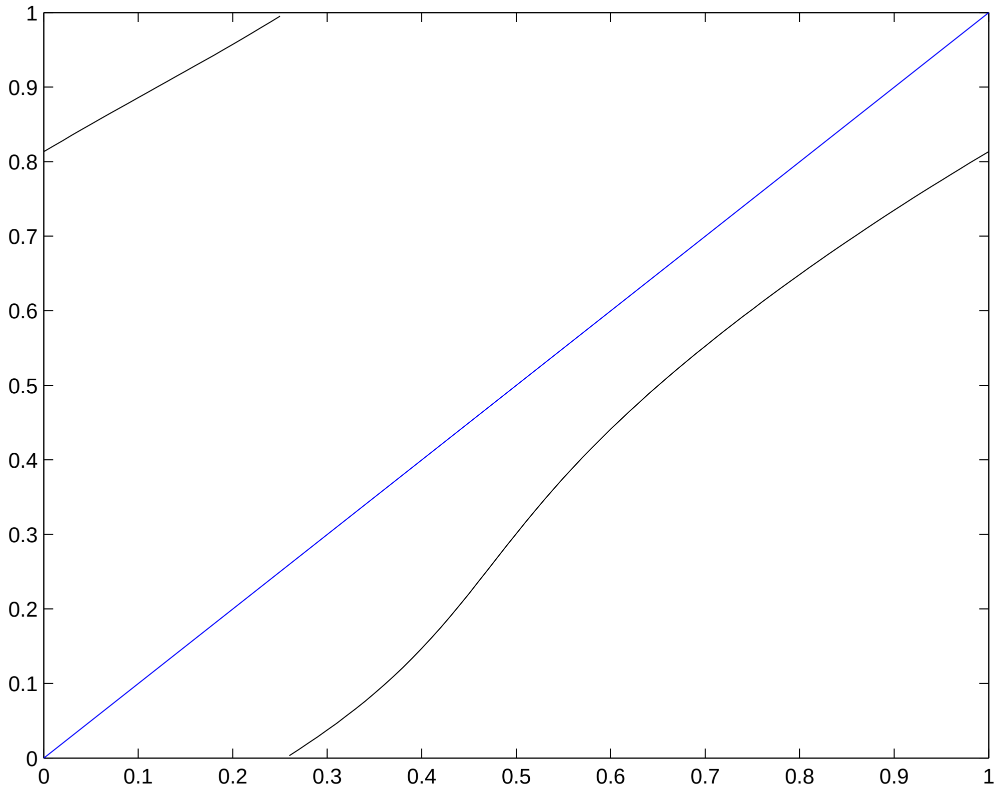
The Half Return Map H

H: flow from $x = 2$ to $x = 1$, jump and apply symmetry

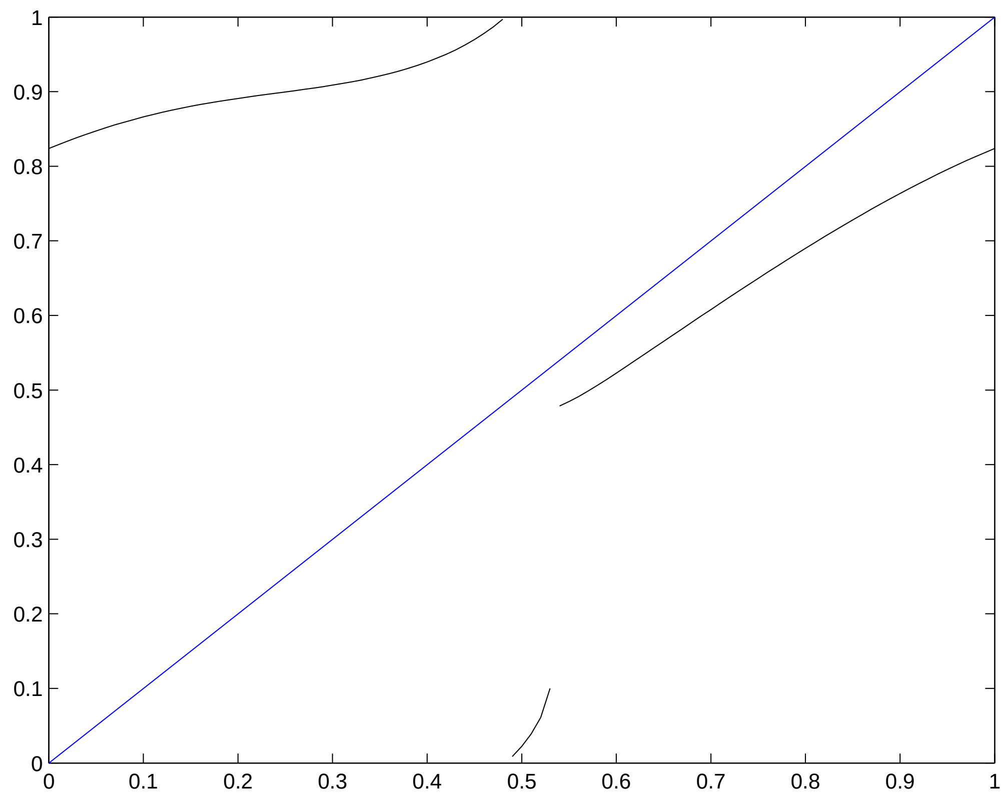
- Fixed points give symmetric periodic orbits with only two jumps
- Three parameter regimes
  - $a < 1$: circle diffeomorphisms - invariant tori
  - $1 < a < 2$: discontinuous monotone maps of circle into arc
  - $2 < a$: discontinuous maps of circle into arc with turning points
- Conjecture: at most 3 fixed points



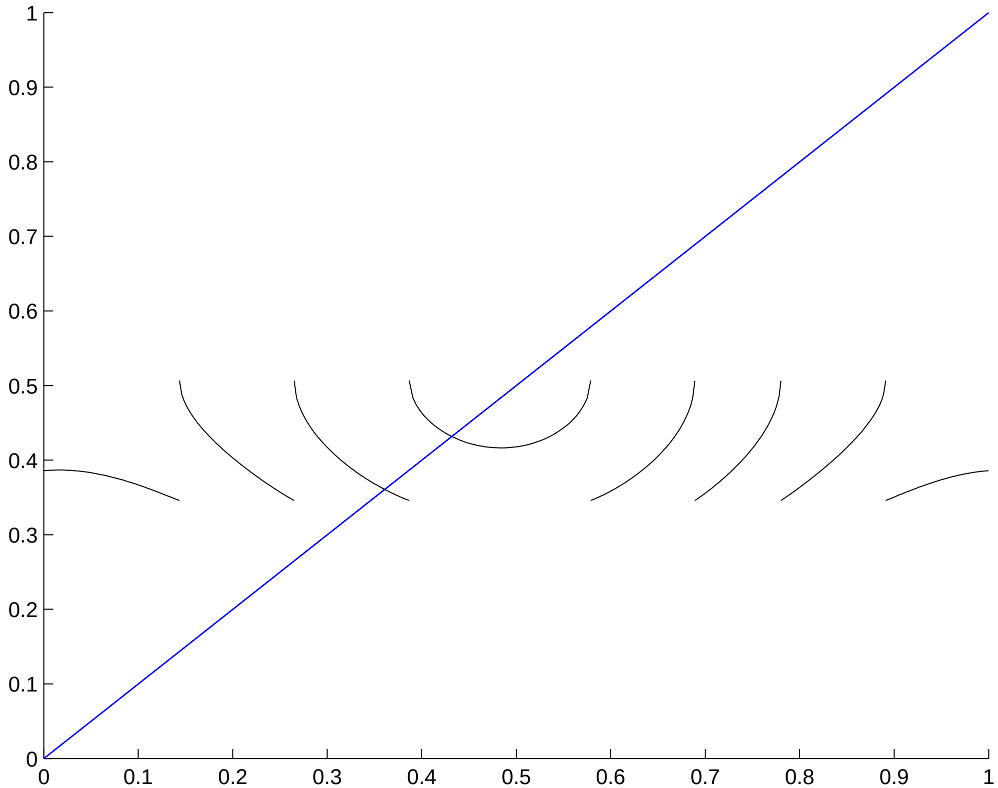
Forced van der Pol slow flow: critical trajectories



Forced van der Pol half-return map: $a = 0.5 \; \omega = 1$



Forced van der Pol half-return map: $a = 1.5 \; \omega = 1$

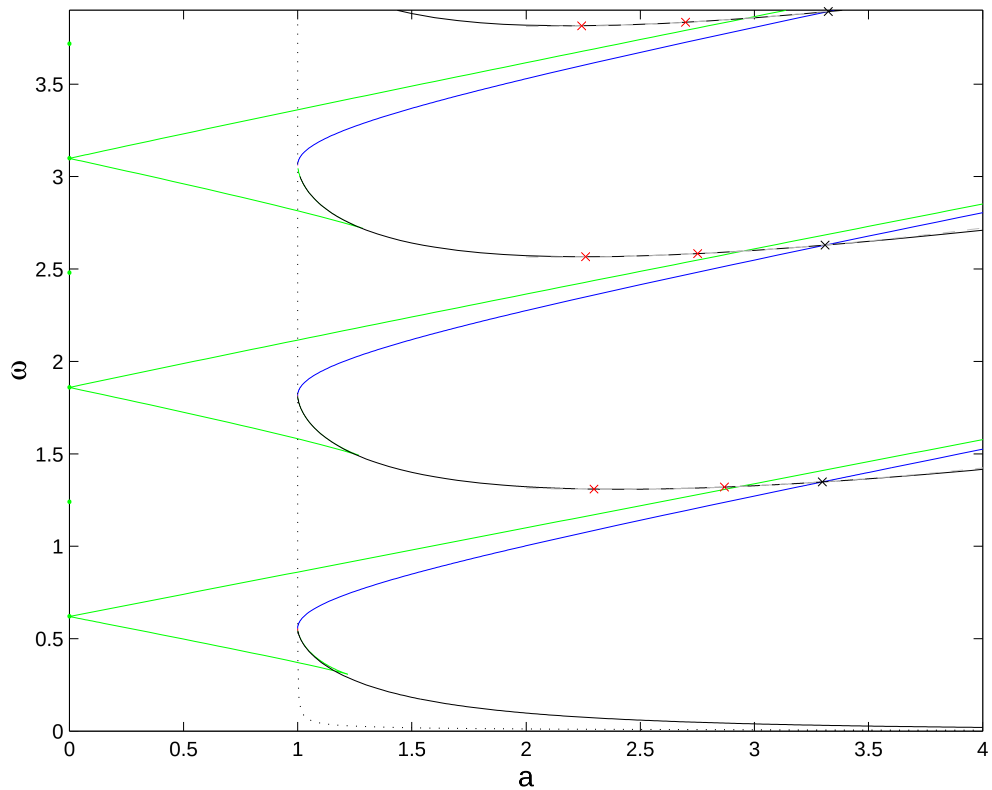


Forced van der Pol half-return map: $a = 20 \; \omega = 5$

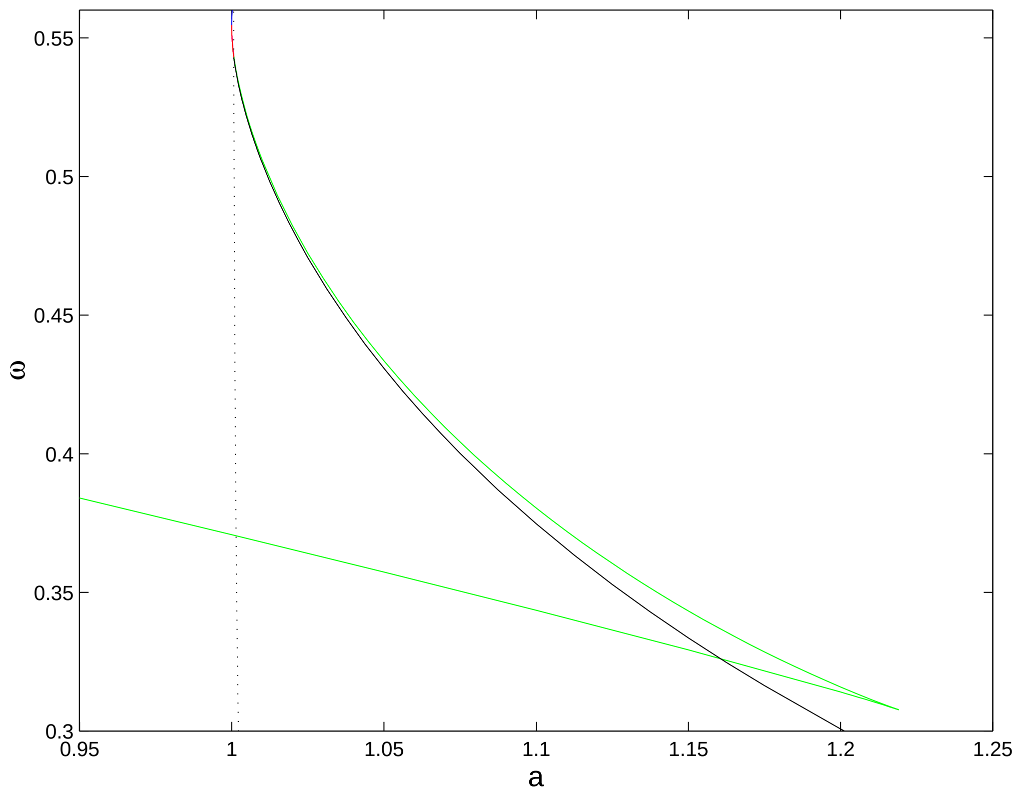
Return Map Bifurcations

Repetitive families indexed by **circuit number**

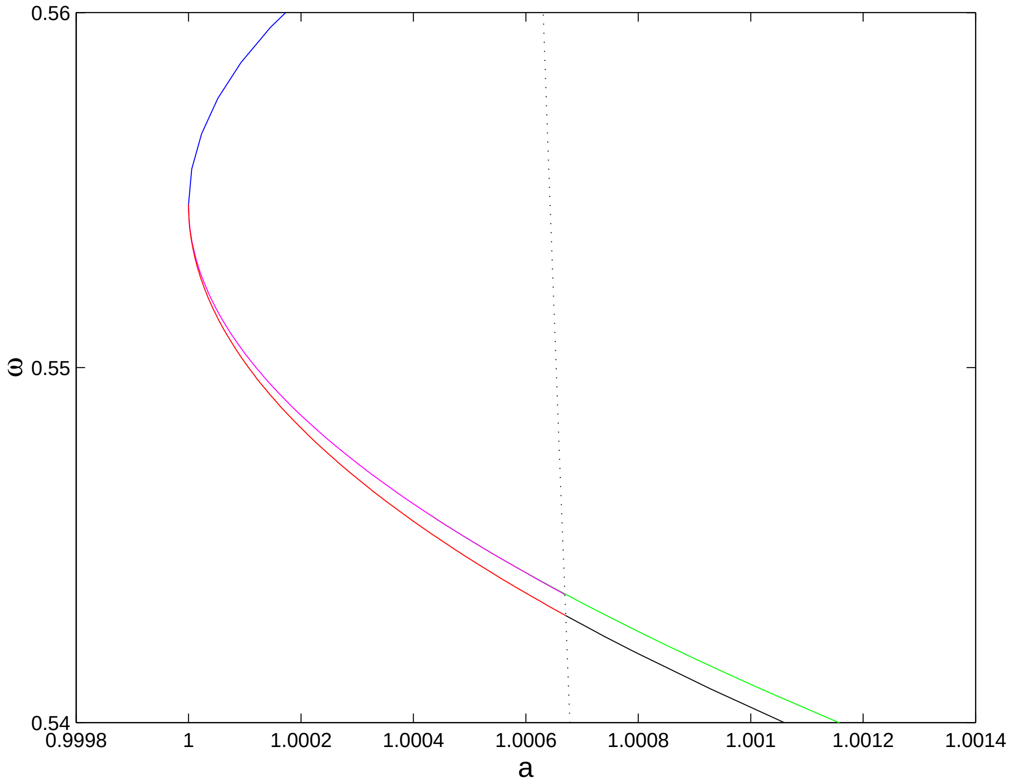
- Homoclinic bifurcations at points of discontinuity for return map
  - Left: saddle jumps to first intersection of stable manifold with $x = 2$
  - Right: p_{1u} jumps to intersection of stable manifold with $x = 2$
- Saddle-node bifurcations of return map
  - Types max and min: local maximum/minimum of $H(x) - x$
- Nodal homoclinic bifurcations where node jumps to intersection of its strong stable manifold with $x = 2$
- Heteroclinic bifurcation where node jumps to stable manifold of saddle



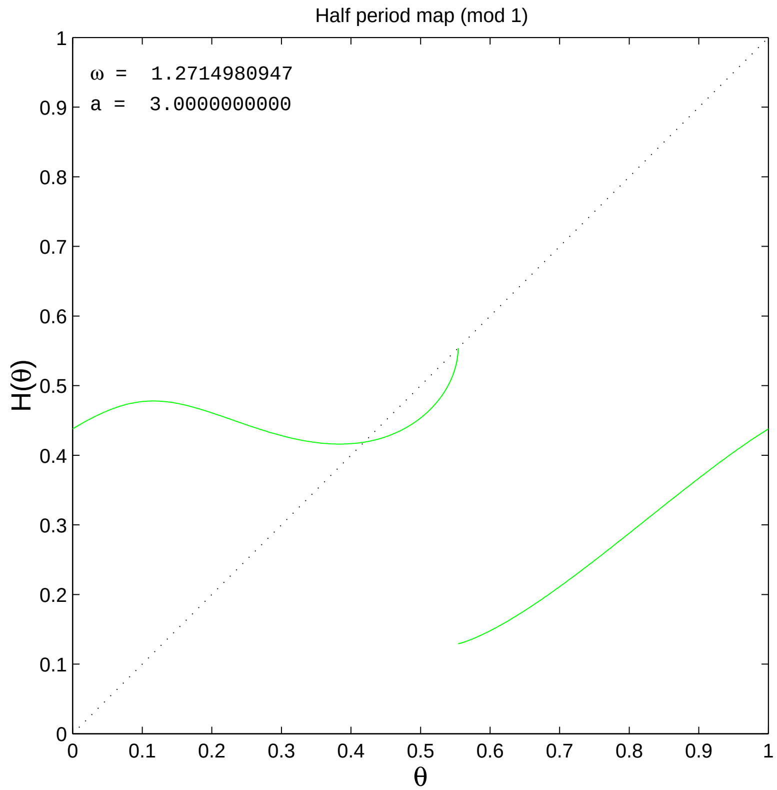
Bifurcation diagram of forced van der Pol slow flow



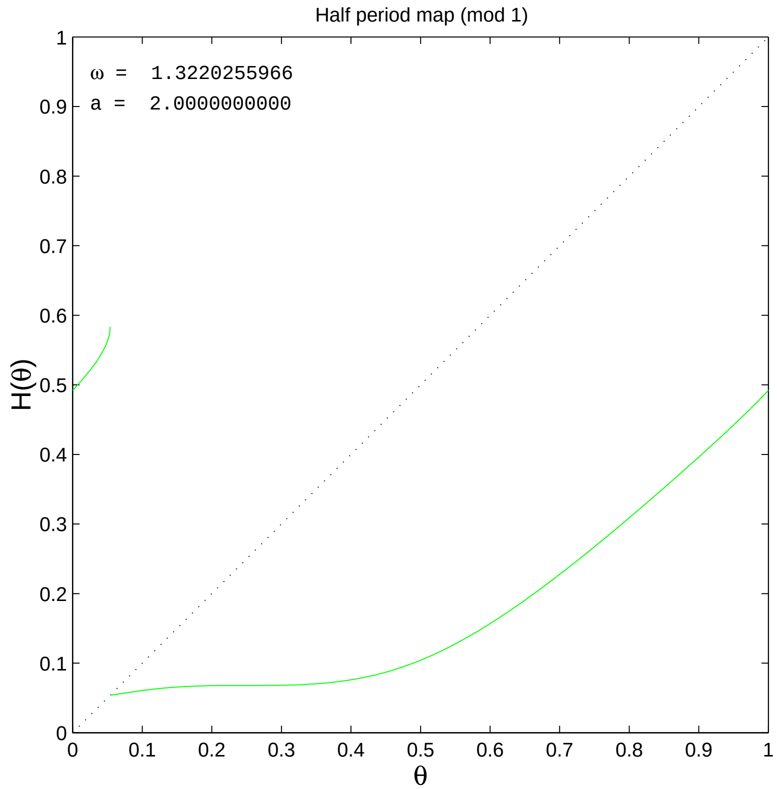
Bifurcation diagram of forced van der Pol slow flow: a cusp of saddle-nodes



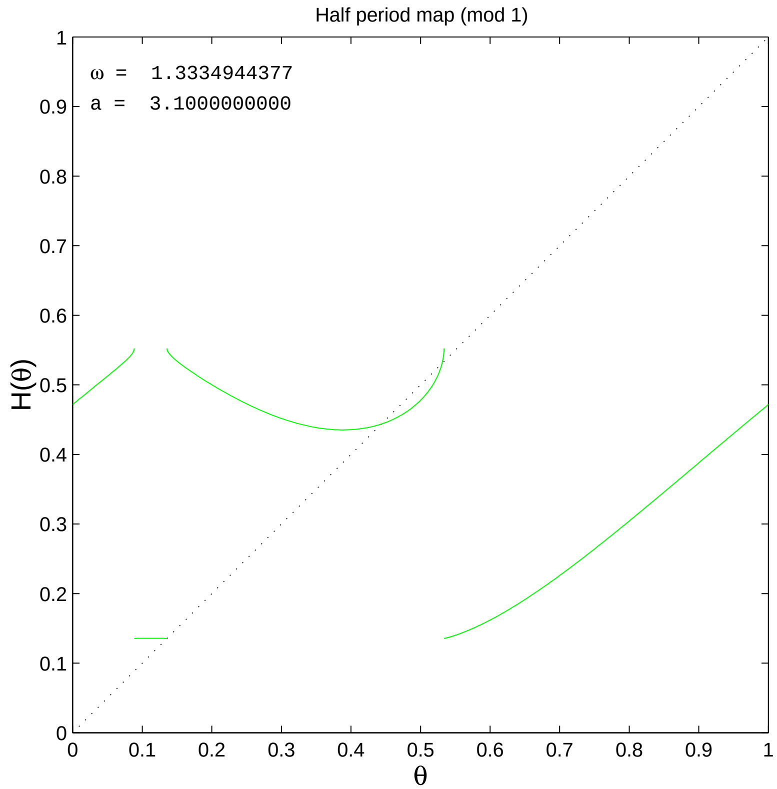
Bifurcation diagram of forced van der Pol slow flow: near a is one



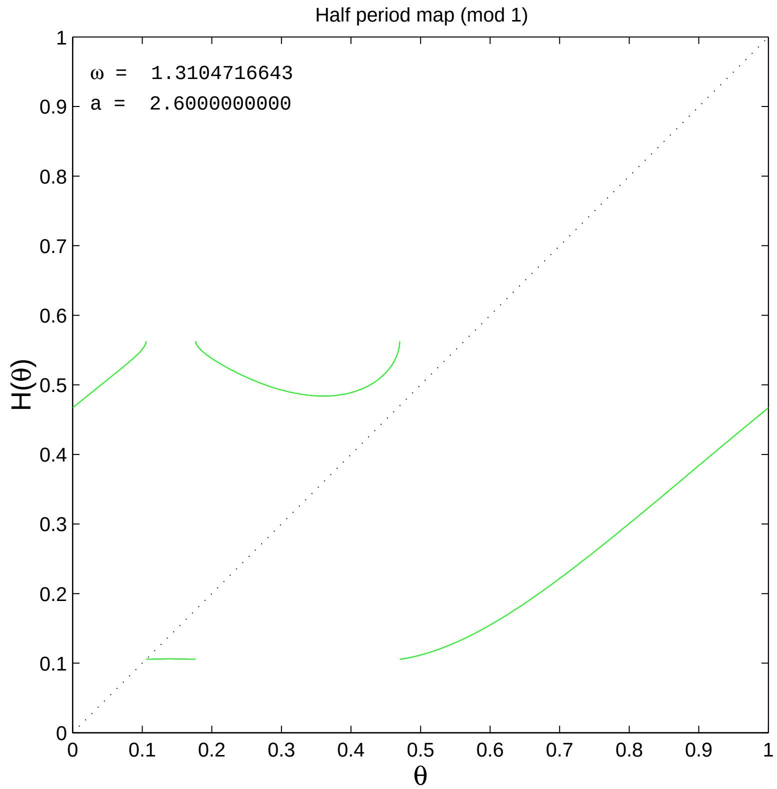
Forced van der Pol half-return map at left homoclinic bifurcation:



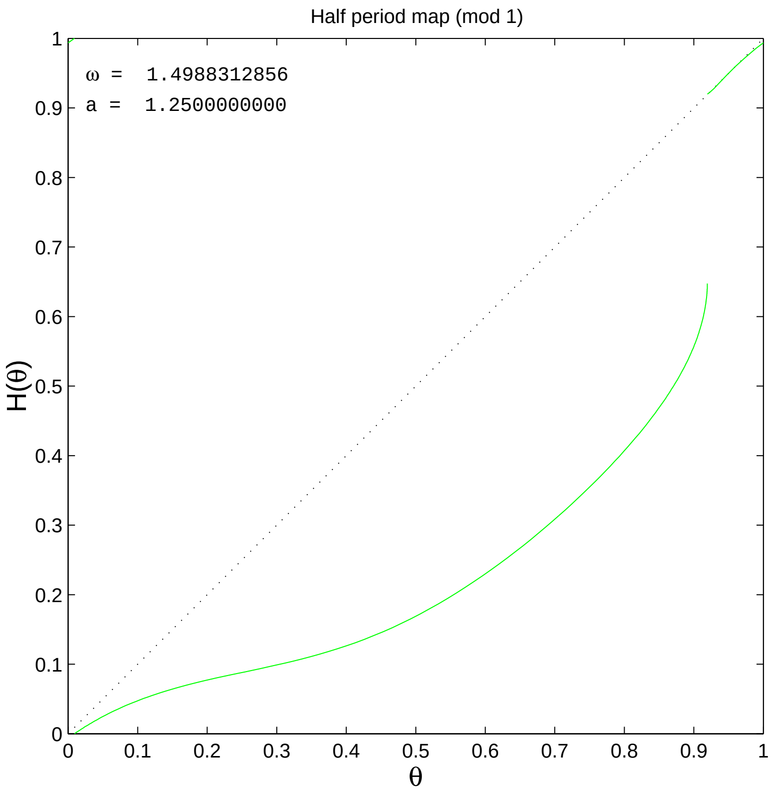
Forced van der Pol half-return map at right homoclinic type bifurcation of type 1



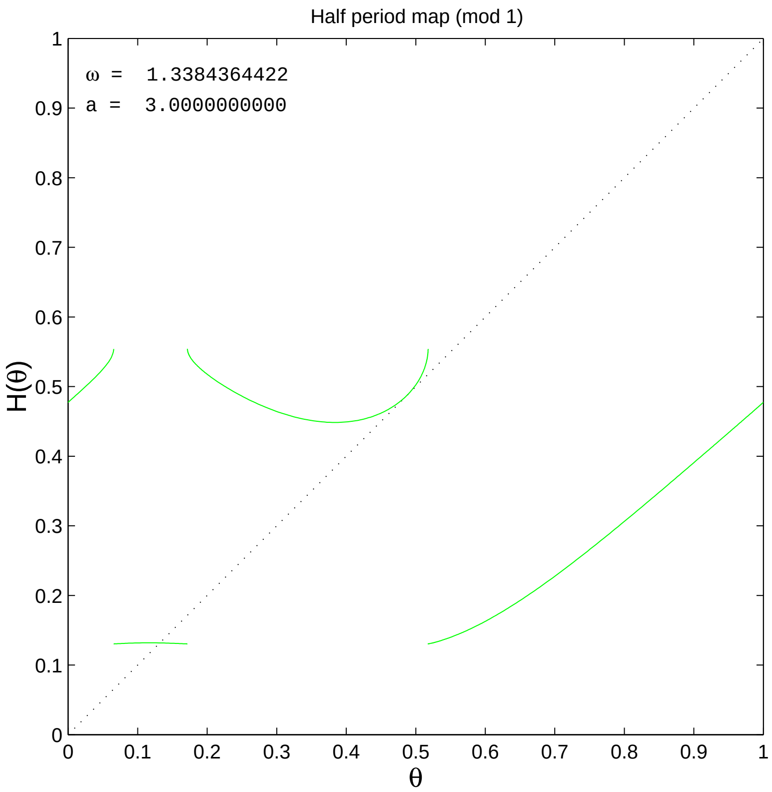
Forced van der Pol half-return map at right homoclinic bifurcation of type 3



Forced van der Pol half-return map at right homoclinic bifurcation of type 2



Forced van der Pol slow flow near max saddle-node

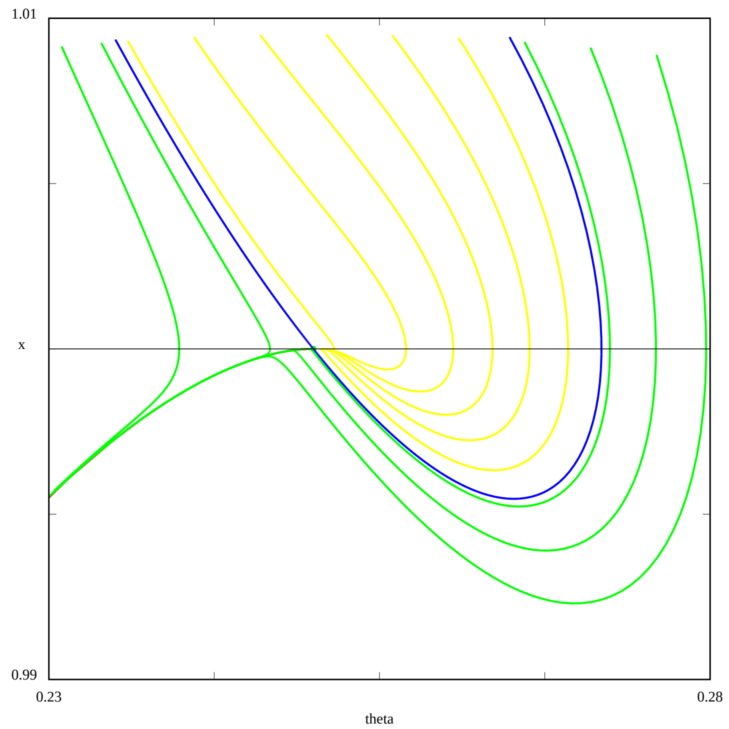


Forced van der Pol slow flow near min saddle-node

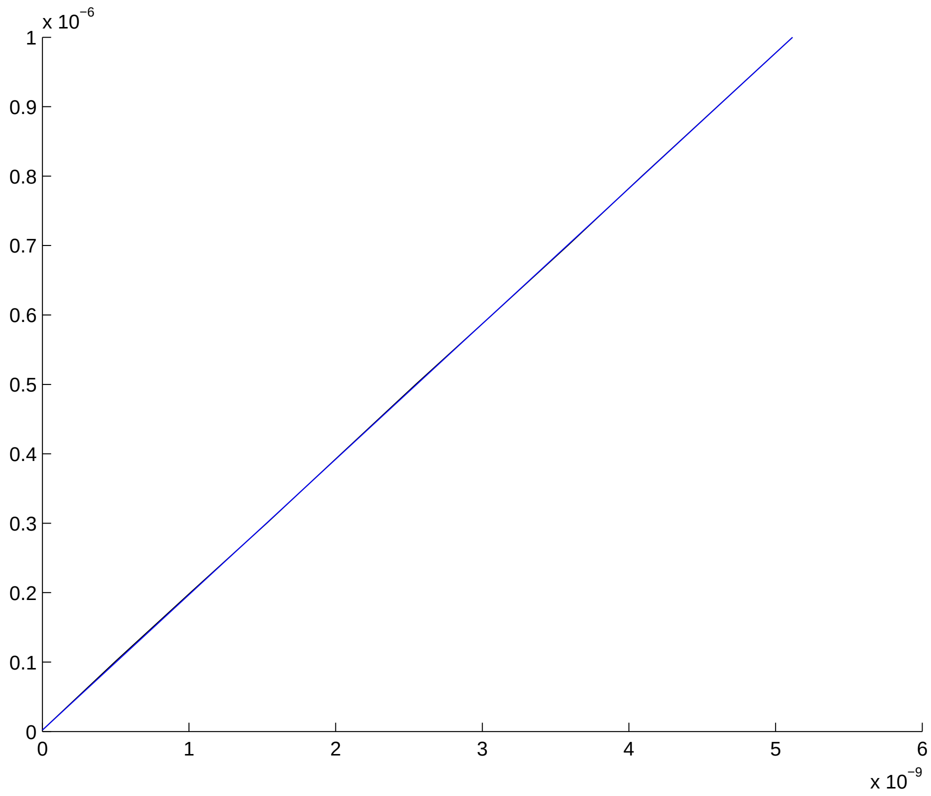
Return Map Codimension Two Bifurcations: Fixed Points

From the bifurcation diagram

- Saddle-node equilibrium with homoclinic trajectory
  - $a = 1$
  - Weak stable orbits jump before reaching saddle-node point
- Cusp: weakly unstable
- Transversal crossings of codimension one bifurcations
  - saddle-nodes with left homoclinics
  - left and right homoclinics

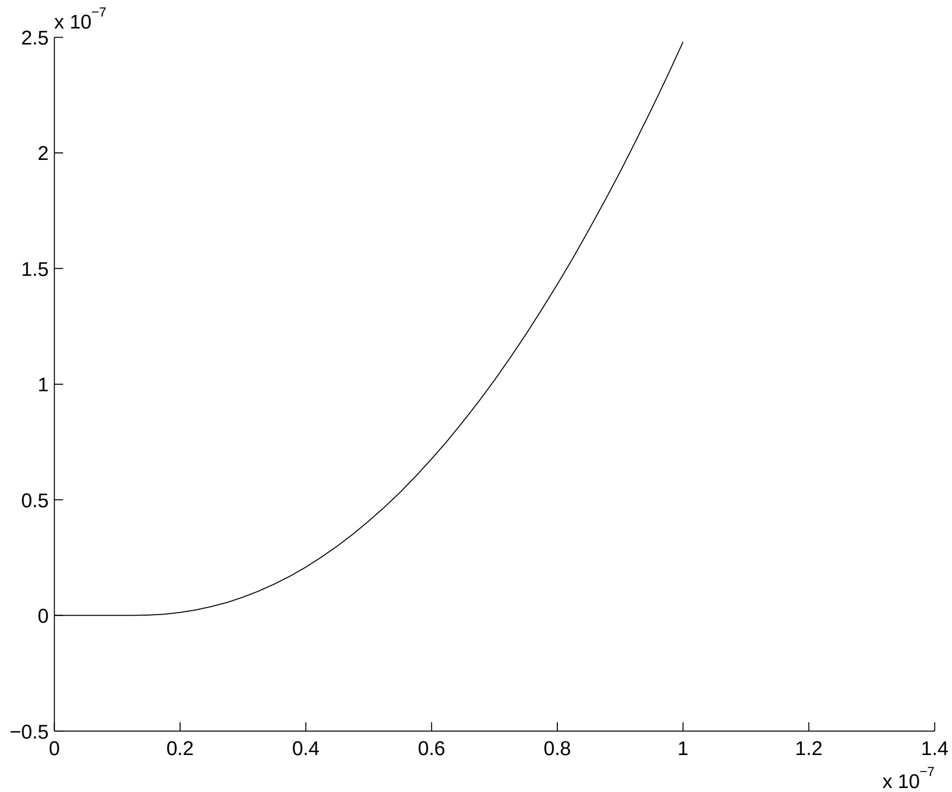


Forced van der Pol flow near saddle-node point



Forced van der Pol half-return map at saddle-node

$$a = 1\ \omega = 1.6$$



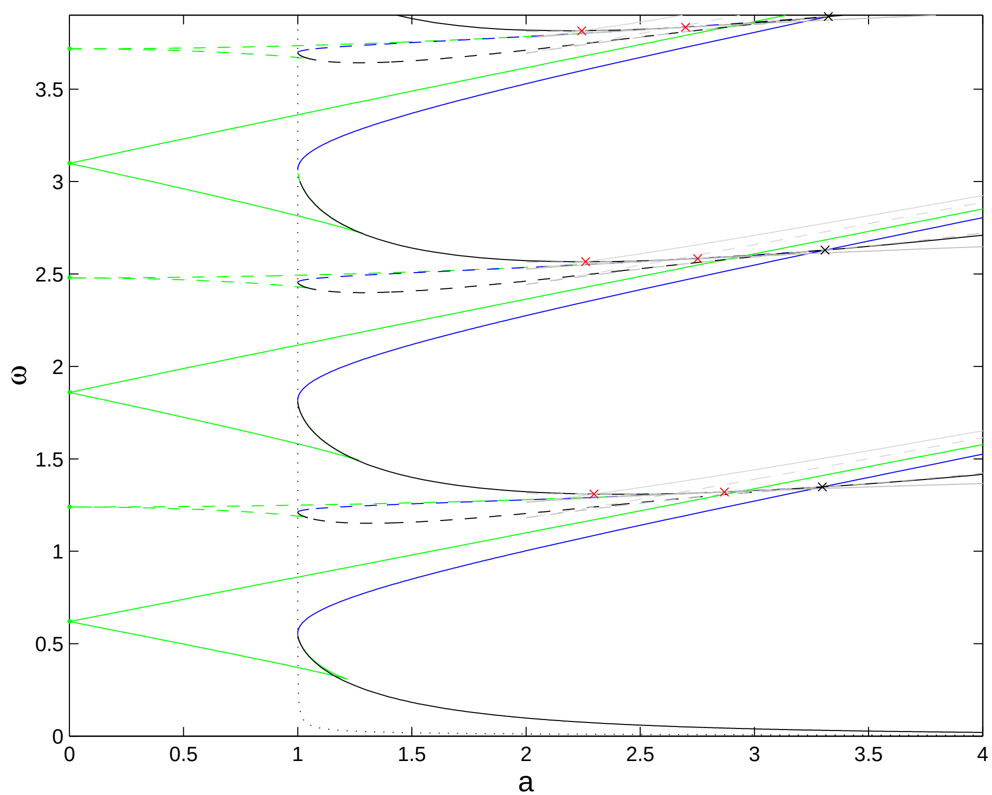
Forced van der Pol half-return map near folded saddle:

$$a = 1.00005 \ \omega = 1.6$$

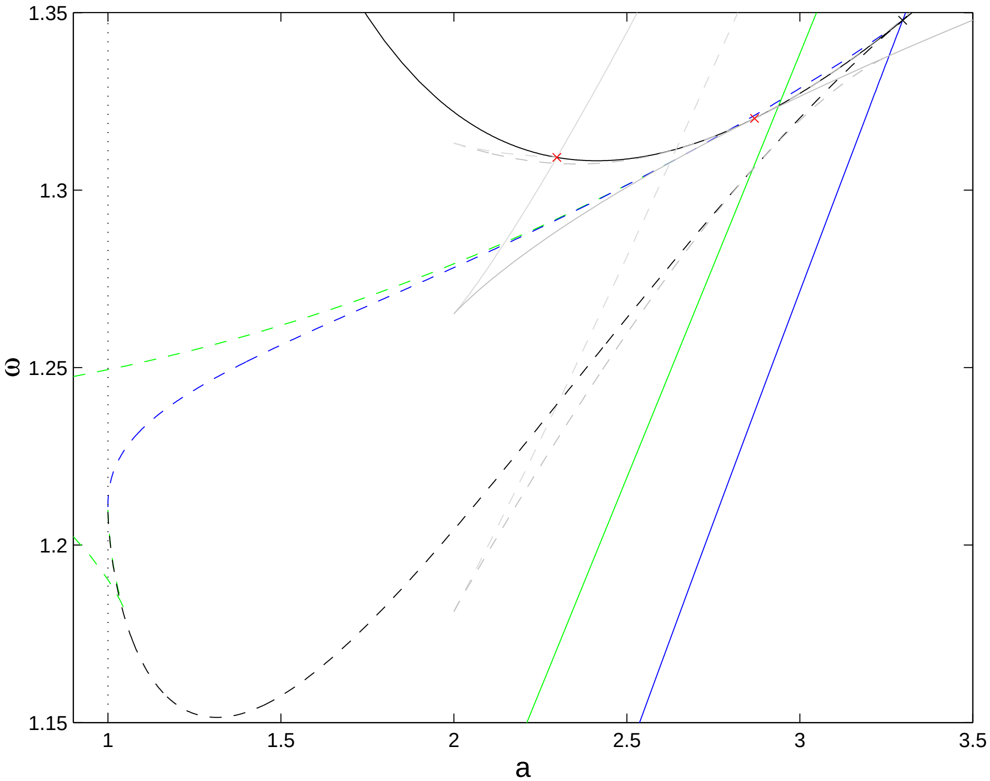
Bifurcations of Period 2 Orbits

Add period 2 bifurcations to bifurcation diagram

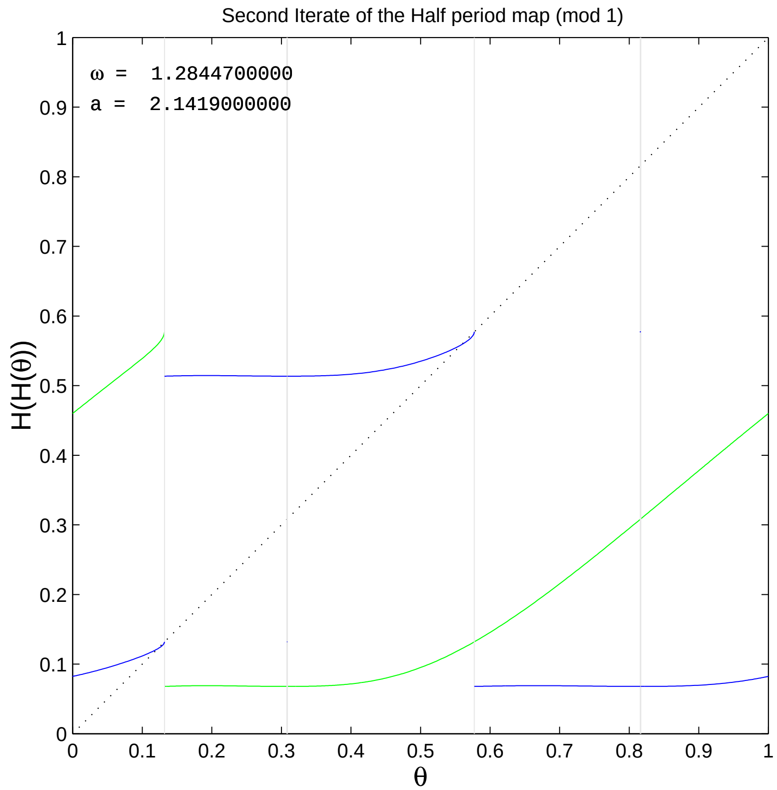
- Many similarities with fixed point bifurcations
- New types of codimension 2 homoclinic bifurcations
- Interaction with codimension 2 bifurcation of fixed points: transversal crossing of right and left homoclinic curves
- Regions with chaotic invariant set



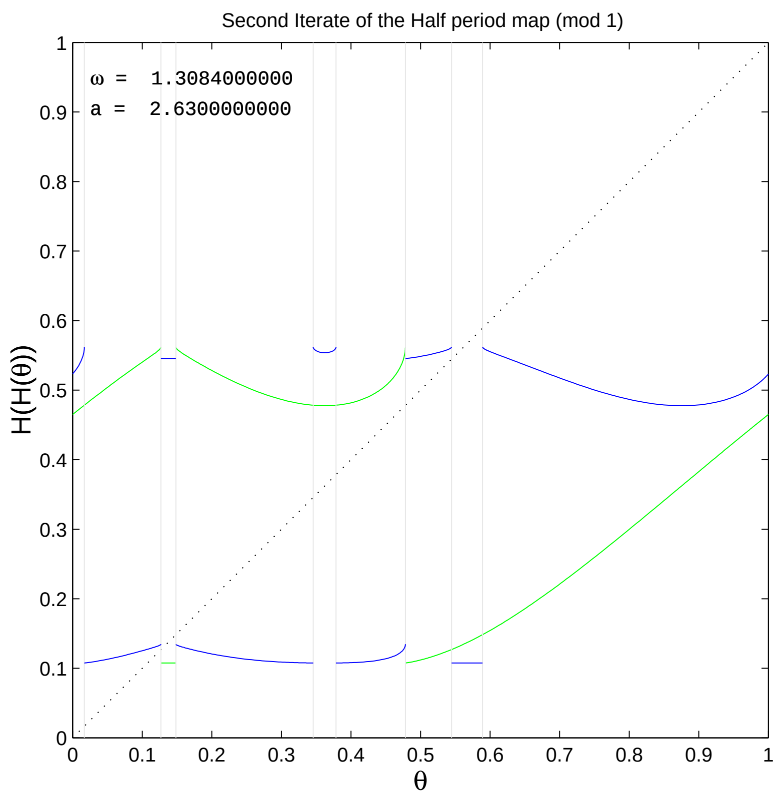
Bifurcation diagram with period 2 oribts



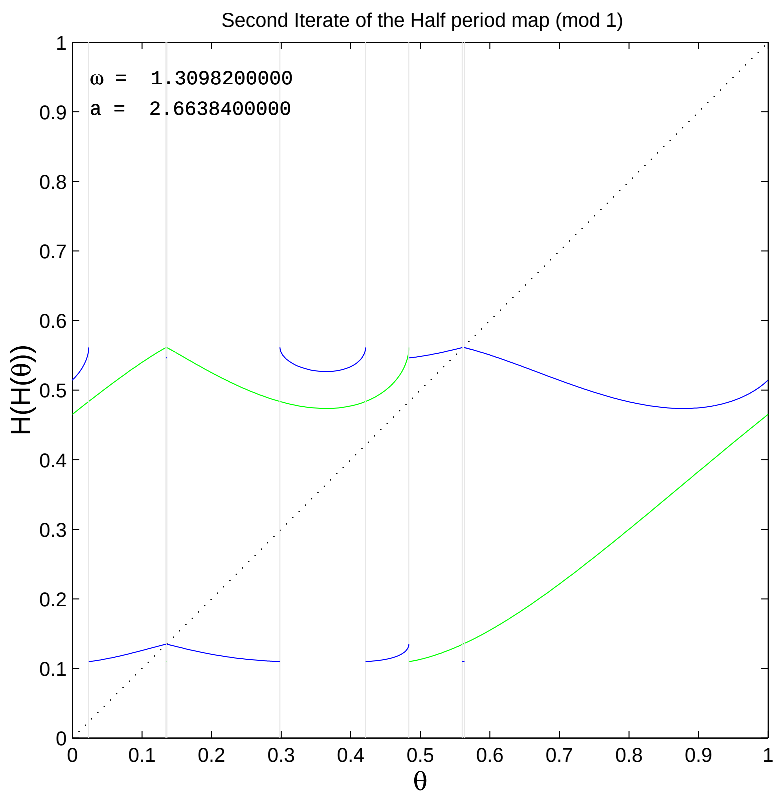
Bifurcation diagram: period 2 orbit details



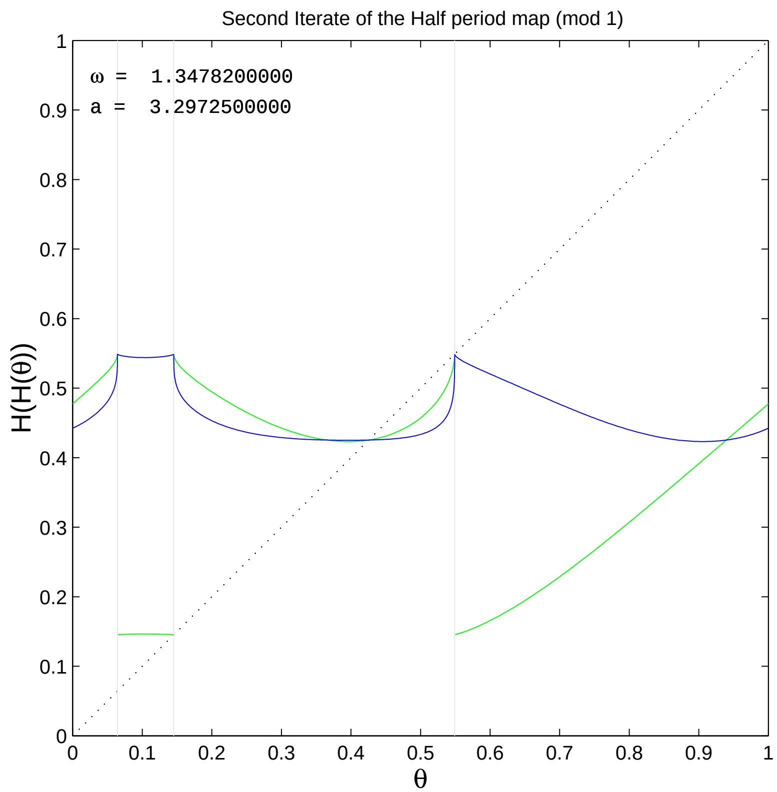
Half return map and second iterate along homoclinic curve



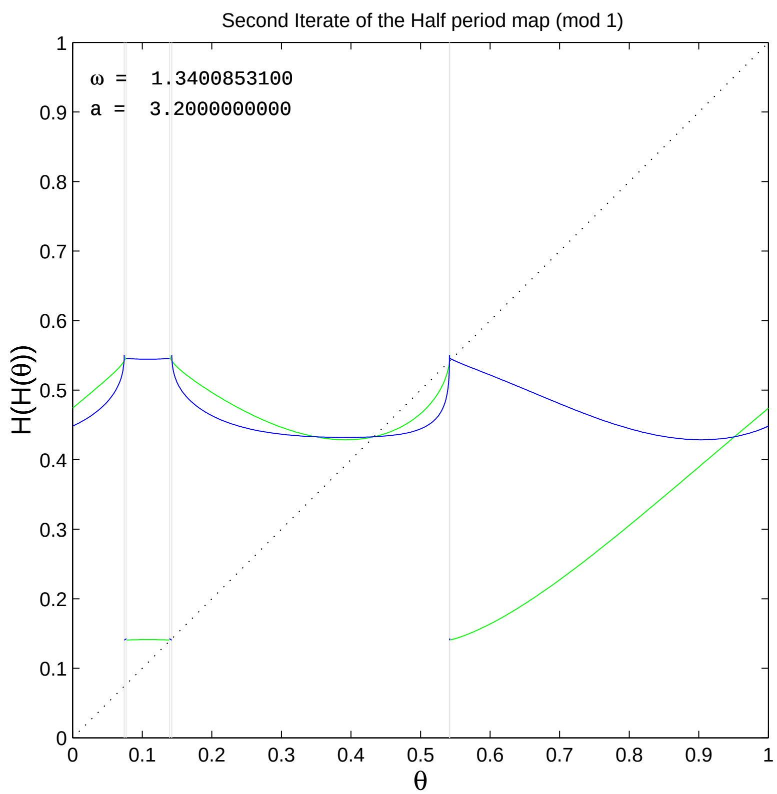
Half return map and second iterate along homoclinic curve



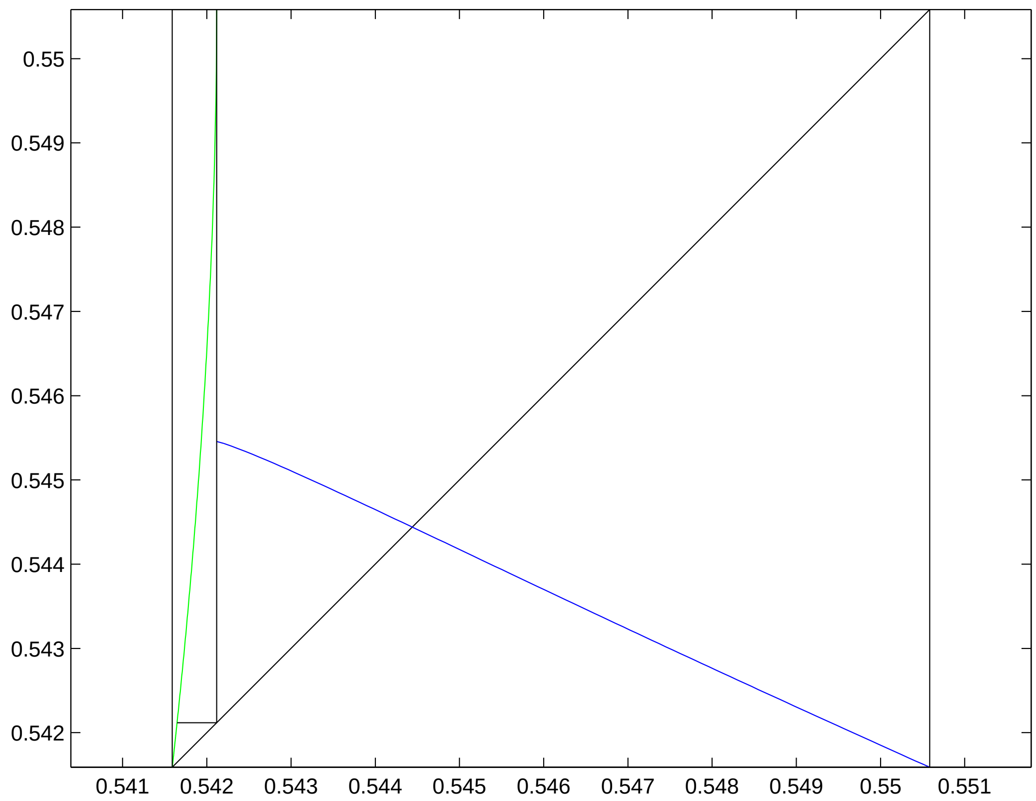
Half return map and second iterate along homoclinic curve



Half return map and second iterate at intersection of homoclinic curves



Half return map and second iterate with chaos



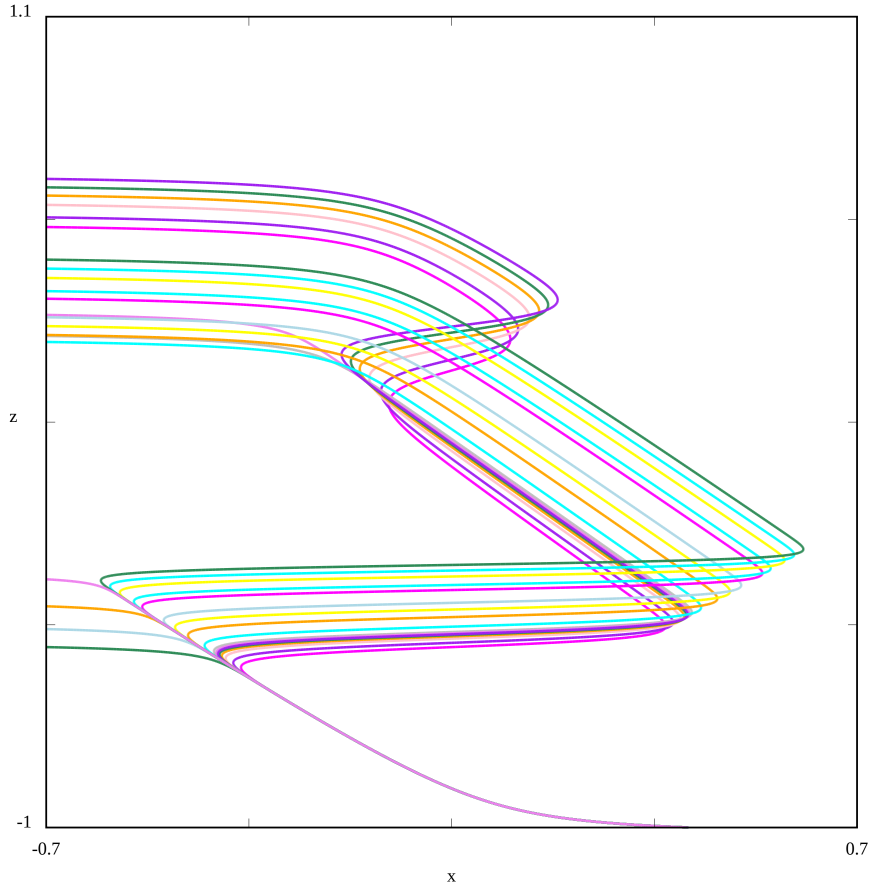
Chaotic return map in small interval

Bifurcations of Forced van der Pol

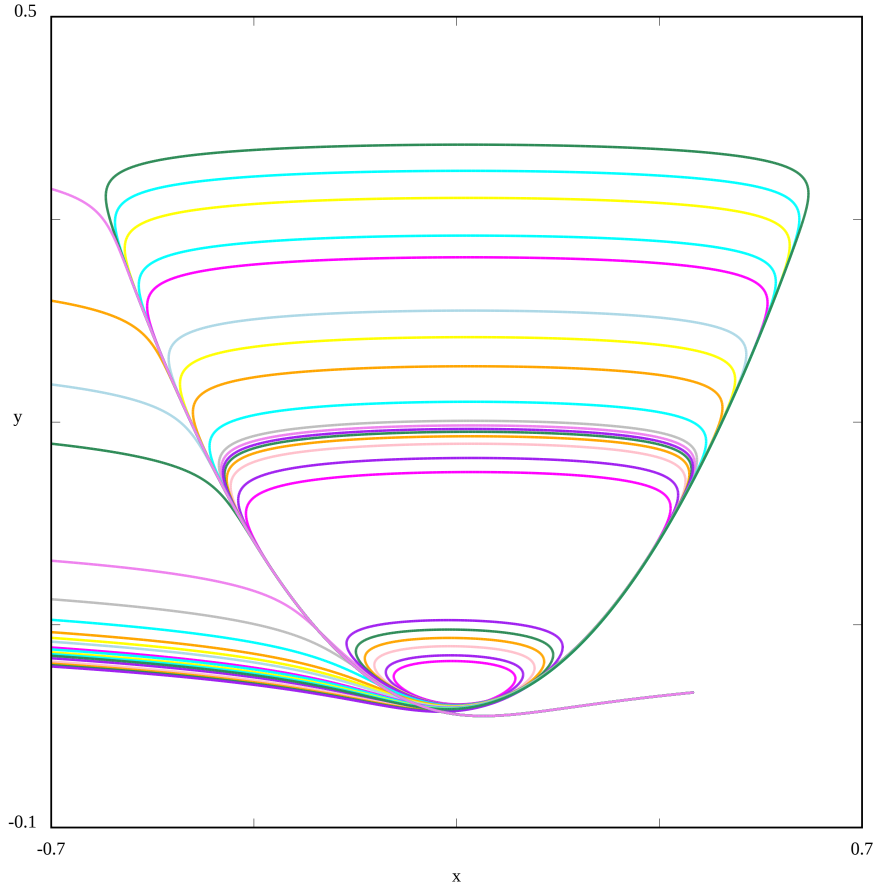
From the singular limit back to the slow-fast system

- Uniformization and asymptotic analysis of folded singularities
- Canards are trajectories that flow onto unstable sheet of critical manifold
- Benoit analysis of folded saddles
- Canards at folded nodes have not been characterized

$$\begin{aligned} \varepsilon \dot{x} &= y - x^2 \\ \dot{y} &= az + bx \\ \dot{z} &= 1 \end{aligned}$$



Folded node trajectories: x vs. z

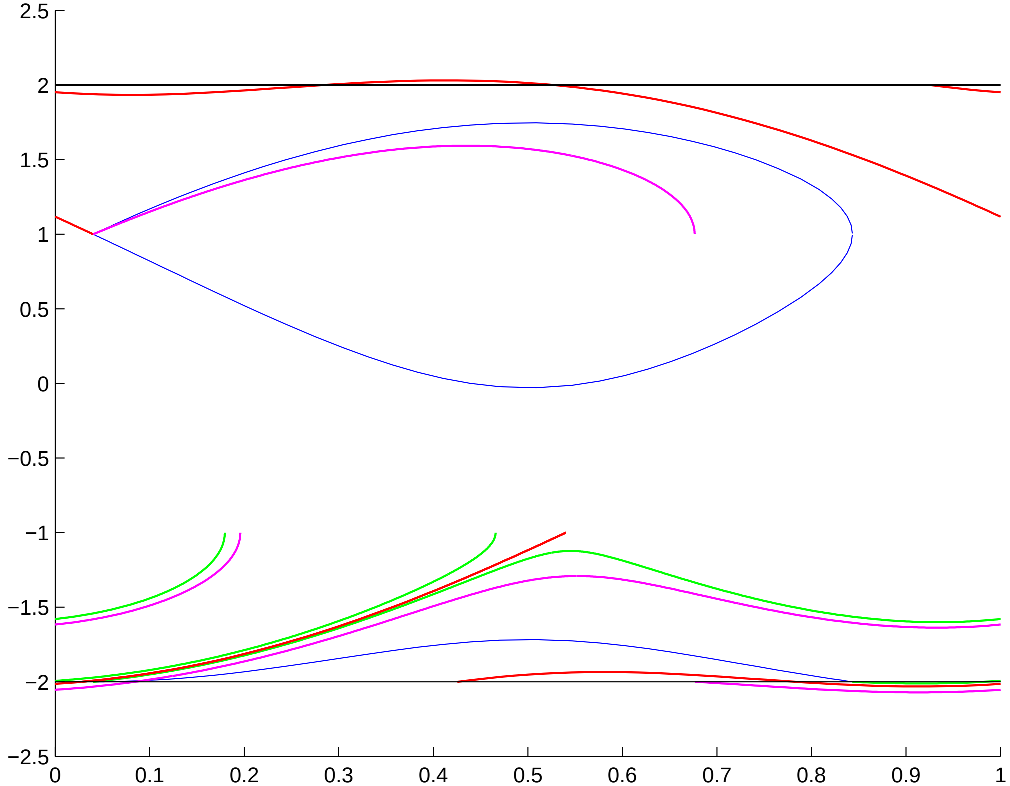


Folded node trajectories: x vs. y

Canards in the Forced van der Pol System

Canards beginning at folded saddles

- Follow (un)stable manifold of folded saddle to a jump point
- Two families of jumps parallel to x axis : up and down
- Multiple canards: jump lands on stable manifold of a folded saddle
- Construct canard return map with "horseshoe": two stable periodic orbits and hyperbolic solenoid
- Circuit numbers of rectangles in construction differ



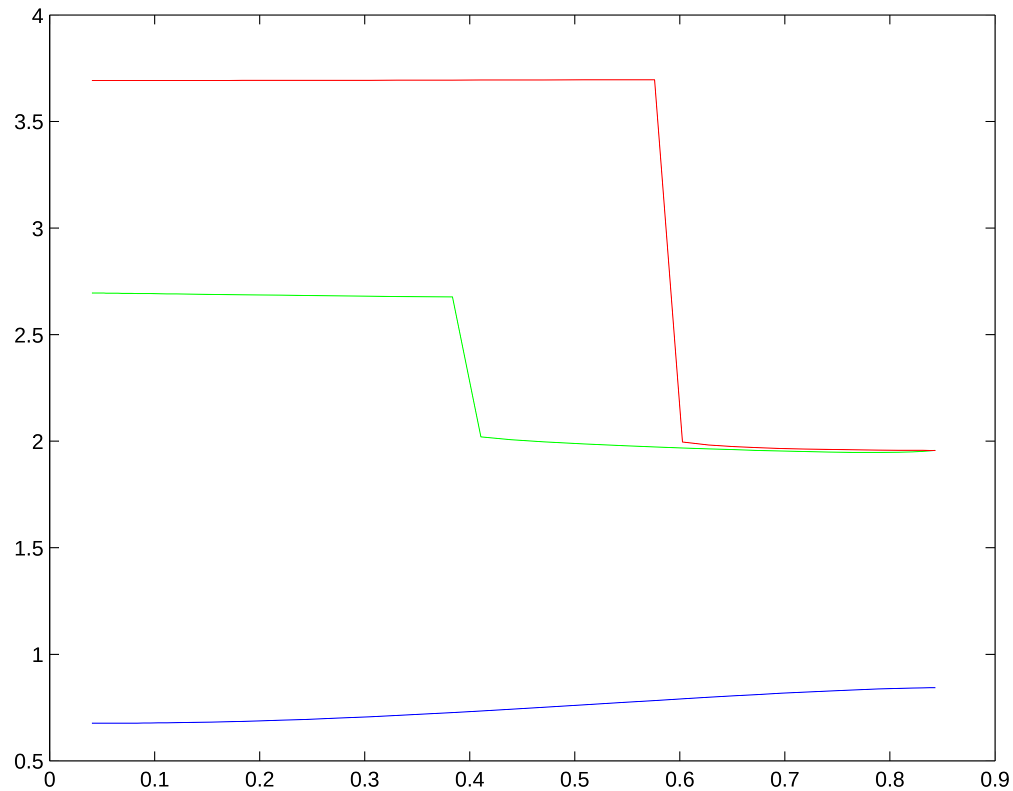
Canard location in forced van der Pol: $a = 4\ \omega = 1.55$

Red: stable manifolds

Magenta: unstable manifold (extended past jump)

Green: trajectories of tin points

Blue: canard and projections



Return to $x = 1$ from canards in forced van der Pol: $a = 4\ \omega = 1.55$

Blue: first return from jumps up

Red: second return from jumps up

Green: first return from jumps down

Generic Relaxation Oscillations

Starting point: slow-fast decompositions

- Classification of generic folds: uniformization
- Focus upon stable slow manifolds and ones with a single unstable direction
- Determine junctions of generic trajectories in generic systems

Degenerate Decompositions: Codimension 1

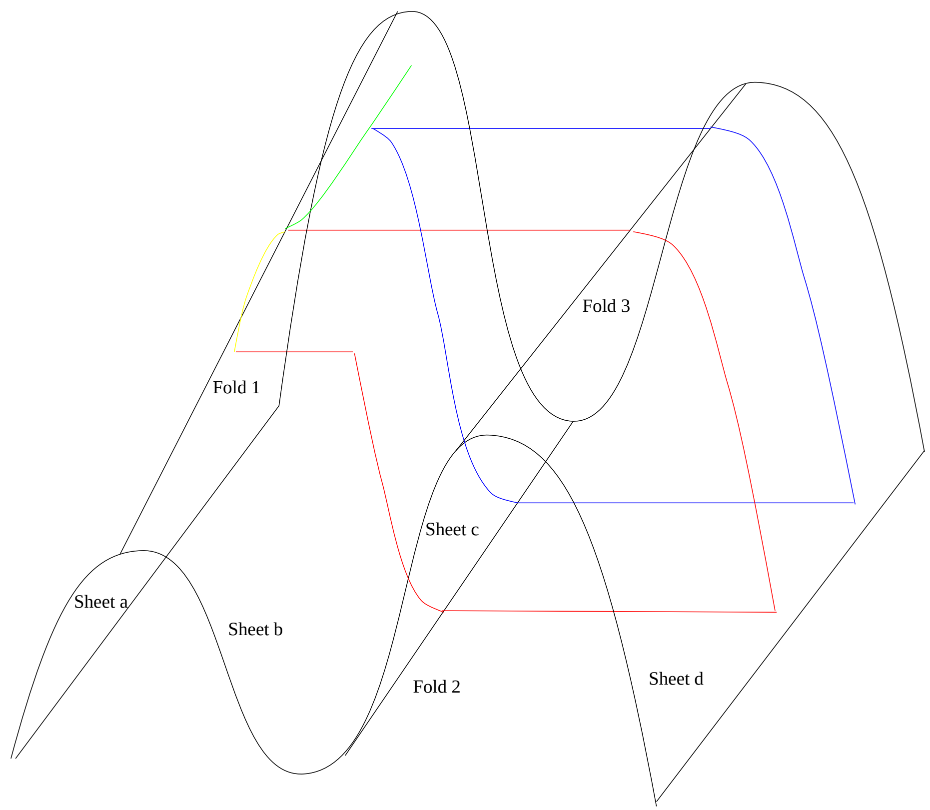
Degeneracies encountered in generic one parameter families of relaxation oscillations beginning at stable orbit

- in-fold $\rightarrow$ out-fold
- folded saddle
- saddle-initiated canards
- Hopf bifurcation at folds
- initiation at cusp
- in-fold $\rightarrow$ in-fold
- jump maps an in-fold non-transversally to slow flow on sheet of slow manifold at the end of the jump

Geometric Models & Asymptotic Analysis

Case of in-fold $\rightarrow$ out-fold

- Family of relaxation oscillations in which junction from fast segment to slow segment reaches an out-fold
- Return map is composition of transitions along slow and fast segments and across jumps
- Model system by normal forms in each region
- Special attention to transition past degeneracy on fold 1
- Canard formation at degeneracy
- Several cases: chaos is possible
- Bifurcations occur as canard formation begins



Degenerate decomposition: in-fold $\rightarrow$ out-fold

Bifurcations of Relaxation Oscillations

Analysis of bifurcations subsidiary to degenerate decompositions

- Examine each type of degenerate bifurcation
- Study uniformizations and asymptotic expansions at degeneracies
- Build models for relaxation oscillations encountering degeneracies
- Maximal and multiple canards lead to hierarchy of exponentials
- Develop algorithms based upon asymptotic analysis

Initial focus upon classical example: forced van der Pol equation