Quantum Computer Algorithms

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Crash Course on Computational Complexity

- Computational Complexity
- Computing Models
- Some notation
- Uniformity

Computational Complexity

- We usually measure the amount of resources (e.g. time, space, gates) used by an algorithm as a function of the input size.
- E.g. The grade-school algorithm for multiplying two n-bit integers uses $O(n^2)$ time steps. FFT methods use $O(n(\log n)(\log\log n))$ time steps. The best known lower bound is $\Omega(n)$ steps.

"polynomial" cost

- When we say an algorithm uses a polynomial amount of some resource (e.g. time, space, gates, energy), we mean that there is some polynomial p(n) such that the amount of that resource used by the algorithm is in $O(p(n))$
- E.g. we can multiply n-bit numbers in polynomial time.

"polynomial" cost

- If the cost is not bounded above by a polynomial, we say its "super-polynomial"; sometimes people abuse the term "exponential" to mean super-polynomial
- E.g. the best rigorous probabilistic classical algorithm for factoring n-bit numbers uses time in $e^{O(\sqrt{n \log n})}$
- So there is no known polynomial time classical algorithm for factoring

What's so special about polynomials?

- The Strong Church-Turing thesis states that a probabilistic Turing machine can simulate any reasonable algorithmic process with at most a polynomial overhead
- Using polynomial cost as our notion of "efficiency" is very convenient.

Computing Models

- Two commonly used models are the Turing machine model and the circuit model

Turing machines

- Turing machines can take inputs of any size.
- We measure the time complexity of a computation on a Turing machine by the number of steps taken before the TM stops
- The space complexity is the number of tape positions used for the computation
- We usually consider the *worst case* complexity for an input of a fixed size n.

Asymptotic Notation

- A function f(n) is in O(g(n)) if for some constant m there exists a positive constant c such that $f(n) \leq c\ g(n)$ for all $n \geq m$
- A function f(n) is in $\Omega(g(n))$ if for some constant m there exists a positive constant c such that $f(n) \geq c\ g(n)$ for all $n \geq m$
- A function f(n) is in $\Theta(g(n))$ if for some constant m there exists positive constants $c_1 \leq c_2$ such that $c_1\ g(n) \leq f(n) \leq c_2\ g(n)$ for all $n \geq m$

Circuits

- We usually measure the complexity of a circuit C_n by its size, $|C_n|$, which is the number of gates in it.
- We can also measure the depth (or time), and the space (or width).
- Circuits only take a fixed size input. So how can we fairly compare them to Turing machines?

Families of Circuits

- We consider families of circuits $\{C_n\}$ where C_n takes inputs of size n.
- We can, e.g., design a family of multiplication circuits where C_n has size $O(n^2)$ or $O(n \log n \log\log n)$.
- Recall that the description of a Turing machine is finite. Where do we keep an infinite family of circuits?

Families of Circuits

- We have a procedure (e.g. a Turing machine) that generates the circuit diagrams for us
- For the size of the circuit C_n to fairly reflect the complexity of solving a problem on an input of size n, the complexity of generating the circuit must be "reasonable"

Families of Circuits

- The definition of "reasonable" varies depending one what you are trying to prove, but as a bare minimum, we expect the time and space complexities of generating C_n to be at most polynomial in the size of C_n
- For most of the circuits we will encounter, it will be clear that we can efficiently generate C_n given the integer n

Families of Circuits

- A family of circuits that can be efficiently generated is a *uniform family* of circuits
- Non-uniform families of circuits can require exponential resources to construct. It is possible to hide valuable information in the circuit C_n that we might not be able to compute from scratch using poly($|C_n|$) resources. It is not appropriate to use $|C_n|$ as a measure of the complexity of solving a problem "from scratch"

Uniform Families of Acyclic Quantum Circuits

- The computing model we will use for most of this course is *uniform families of acyclic circuits*
- The word "circuit" seems to refer to particular physical implementation of a computer. We will often use the terms "network" or "array of gates" instead.

Quantum Algorithms Overview

- Eigenvalue Estimation lets us factor integers
- Eigenvalue 'kick-back' turns eigenvalue estimation problem into phase estimation problem
- Quantum Fourier Transform and Phase Estimation
- Generalization to finding hidden subgroups
- Finding Hidden Affine Functions

Integer Factorization

- The security of many public key cryptosystems used in industry today relies on the difficulty of factoring large numbers into smaller factors.
- Factoring the integer N into smaller factors can be reduced to the following task:

Given integer a, find the smallest positive integer r so that $a^r \equiv 1 \bmod N$

Simple operator

Since we know how to efficiently multiply by a mod N, we can efficiently implement

$$U_a |x\rangle = |ax\rangle$$

Note that $U_a^r |x\rangle = |a^r x\rangle = |x\rangle$

i.e. $U_a^r = I$

Interesting eigenvalues

If $U_a^r = I$ then the eigenvalues of U_a are of the form $e^{2\pi i \frac{k}{r}}$

$$U_a |\psi_k\rangle = e^{i2\pi\frac{k}{r}} |\psi_k\rangle$$

$$|\psi_k\rangle = \sum_{j=0}^{r-1} e^{i2\pi j\frac{k}{r}} |a^j\rangle$$

Checking the eigenvalue

$$U_a \left|\psi_k\right\rangle = \sum_{j=0}^{r-1} e^{-i2\pi j\frac{k}{r}} U_a \left|a^j\right\rangle$$

$$= \sum_{j=0}^{r-1} e^{-i2\pi j\frac{k}{r}} \left|a^{j+1}\right\rangle = e^{i2\pi\frac{k}{r}} \left(\sum_{j=1}^{r} e^{-i2\pi j\frac{k}{r}} \left|a^j\right\rangle \right)$$

$$= e^{i2\pi\frac{k}{r}} \left(\sum_{j=0}^{r-1} e^{-i2\pi j\frac{k}{r}} \left|a^j\right\rangle \right) = e^{i2\pi\frac{k}{r}} \left|\psi_k\right\rangle$$

Finding r

For most integers k, a good estimate of $\frac{k}{r}$ (with error at most $\frac{1}{2r^2}$) allows us to determine r (even if we don't know k). (using continued fractions)

Where do we get $|\Psi_k\rangle$?

Since most k are good, a random $|\Psi_k\rangle$ suffices. Try $|1\rangle = \sum_{k=0}^{r-1} \frac{1}{\sqrt{r}} |\Psi_k\rangle$

Estimating Random Eigenvalue lets us Factor

In summary:
Factoring large numbers can be reduced to estimating a random eigenvalue of U_a

Must make the "global" phase a "relative" phase

A global phase has no physical significance.

In other words, states that differ only by a global phase are equivalent

$$U\left(\sum_x a_x |x\rangle\right) = \sum_x b_x |x\rangle$$

$$U\left(e^{i\theta}\sum_x a_x |x\rangle\right) = e^{i\theta}\sum_x b_x |x\rangle$$

so $$e^{i\theta}|\Phi\rangle \approx |\Phi\rangle$$

Must make the "global" phase a "relative" phase

A relative phase can affect outcome probabilities
E.g.

$$|0\rangle + e^{i\varphi}|1\rangle \xrightarrow{H} \left(\frac{1+e^{i\varphi}}{2}\right)|0\rangle + \left(\frac{1-e^{i\varphi}}{2}\right)|1\rangle$$

$$p_0 = \cos^2\left(\frac{\varphi}{2}\right)$$

Eigenvalue "kick-back"

We can also efficiently implement

$$c\text{–}U_a|0\rangle|x\rangle = |0\rangle|x\rangle$$

$$c\text{–}U_a|1\rangle|x\rangle = |1\rangle|ax\rangle$$

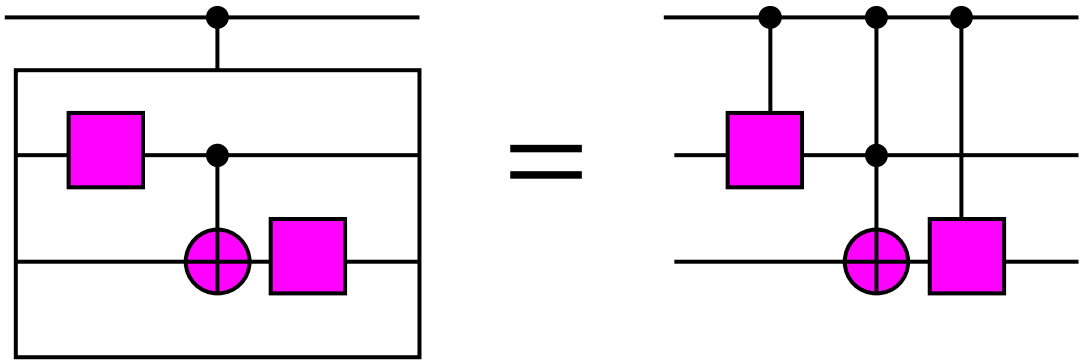
so

$$c\text{–}U_a|0\rangle|\psi_k\rangle = |0\rangle|\psi_k\rangle$$

$$c\text{–}U_a|1\rangle|\psi_k\rangle = e^{2\pi i\frac{k}{r}}|1\rangle|\psi_k\rangle$$

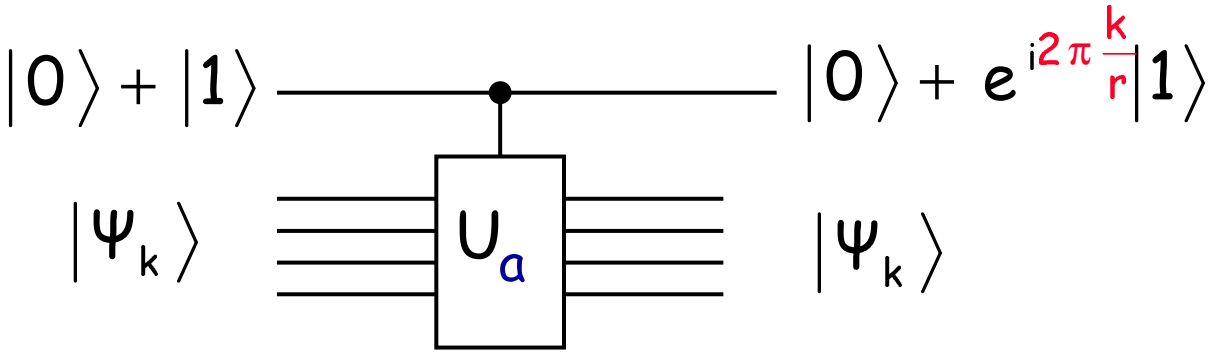
How do we implement c-U?

Replace every gate G in the circuit for with a c-G.
For example,



Eigenvalue kick-back

We can thus efficiently implement

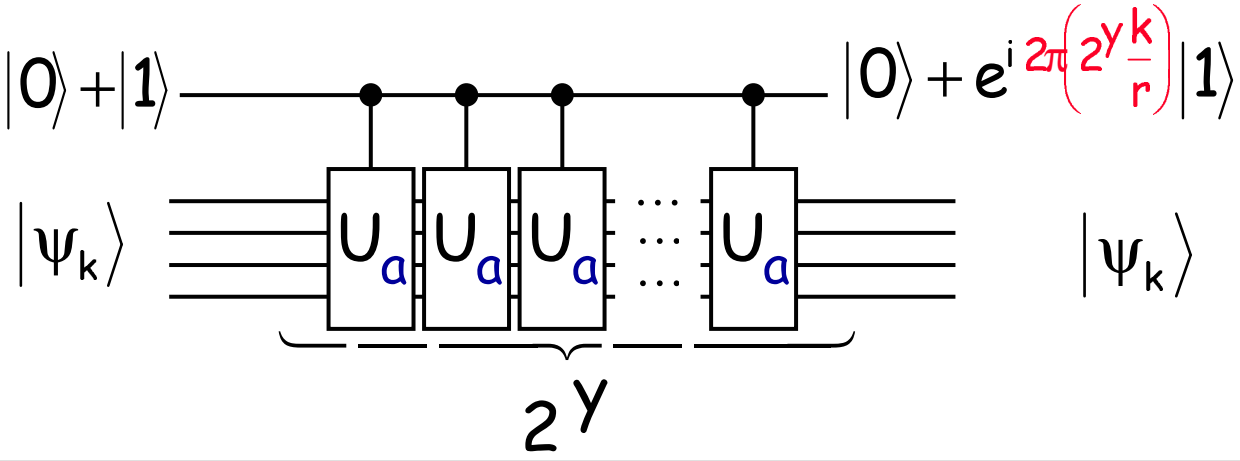


This gives us a relative phase shift of

$\varphi = 2\pi\frac{k}{r}$ in the control qubit

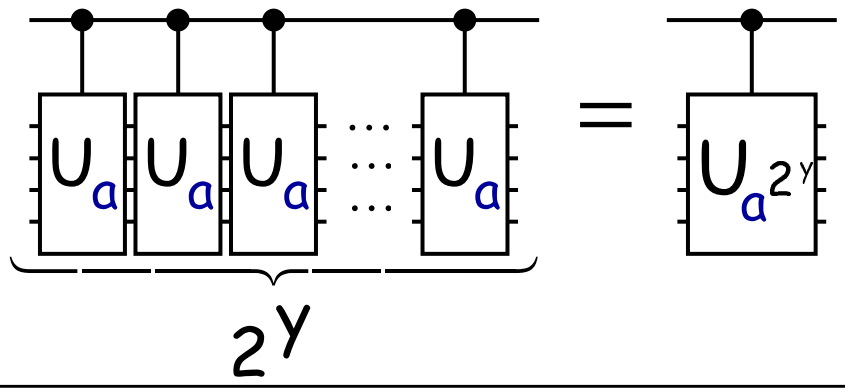
Inefficient exponentiation

We can effect a relative phase shift of $e^{i 2\pi \left(2^y \frac{k}{r}\right)}$



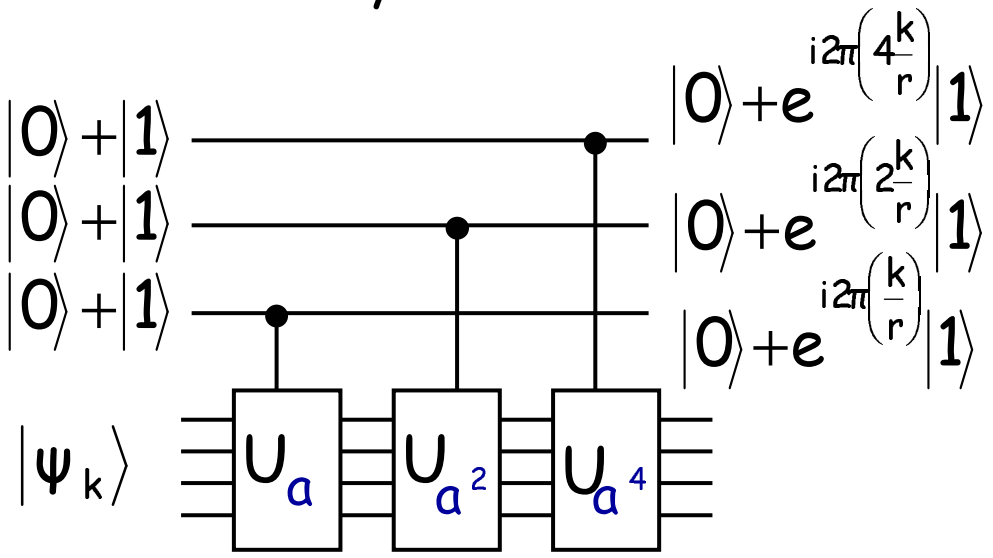
Efficient Exponentiation

But we can also do it **efficiently** by noticing that $U_a^{2^y} = U_{a^{2^y}}$



Reduction to phase estimation

We can efficiently construct



Phase Estimation

Given the qubits

$$\left(|0\rangle + e^{i2\pi\left(\frac{k}{r}\right)}|1\rangle\right)\left(|0\rangle + e^{i2\pi\left(2\frac{k}{r}\right)}|1\rangle\right)\cdots\left(|0\rangle + e^{i2\pi\left(2^j\frac{k}{r}\right)}|1\rangle\right)$$

Estimate $\frac{k}{r}$

Special Case

$$\left(|0\rangle + e^{i(\varphi)}|1\rangle\right) \quad \left(|0\rangle + e^{i(2\varphi)}|1\rangle\right) \quad \left(|0\rangle + e^{i(4\varphi)}|1\rangle\right)$$

Where

$$\frac{\varphi}{2\pi} = \frac{x}{8} = \frac{x_1 x_2 x_3}{8} = \frac{4x_1 + 2x_2 + x_3}{8} = 0.x_1 x_2 x_3$$

Since $e^{i2\pi} = 1$ then we have the state

$$\left(|0\rangle + e^{i2\pi(0.x_1 x_2 x_3)}|1\rangle\right)\left(|0\rangle + e^{i2\pi(0.x_2 x_3)}|1\rangle\right)\left(|0\rangle + e^{i2\pi(0.x_3)}|1\rangle\right)$$

Recall Hadamard transform

$$|b\rangle \xleftrightarrow{H} \left(|0\rangle + (-1)^b |1\rangle\right) = \left(|0\rangle + e^{i2\pi(0.b)} |1\rangle\right)$$

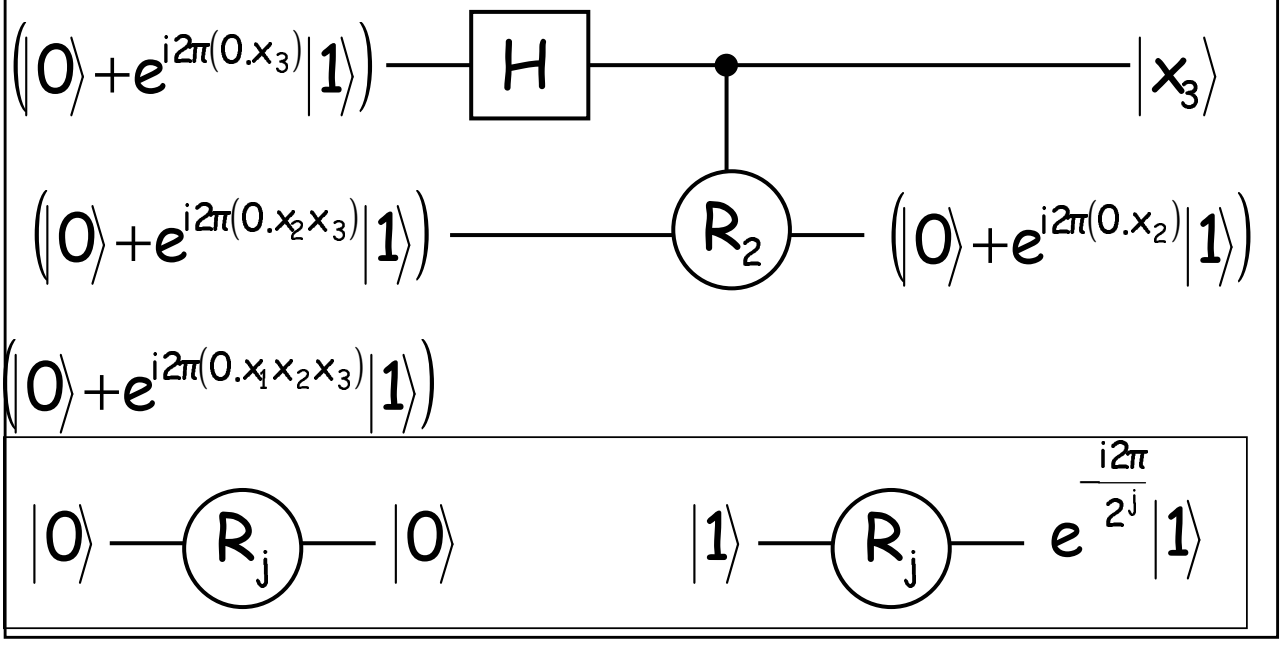
Obvious Phase Estimation Algorithm

$\left(|0\rangle + e^{i2\pi(0.x_3)}|1\rangle\right)$ — H — $|x_3\rangle$

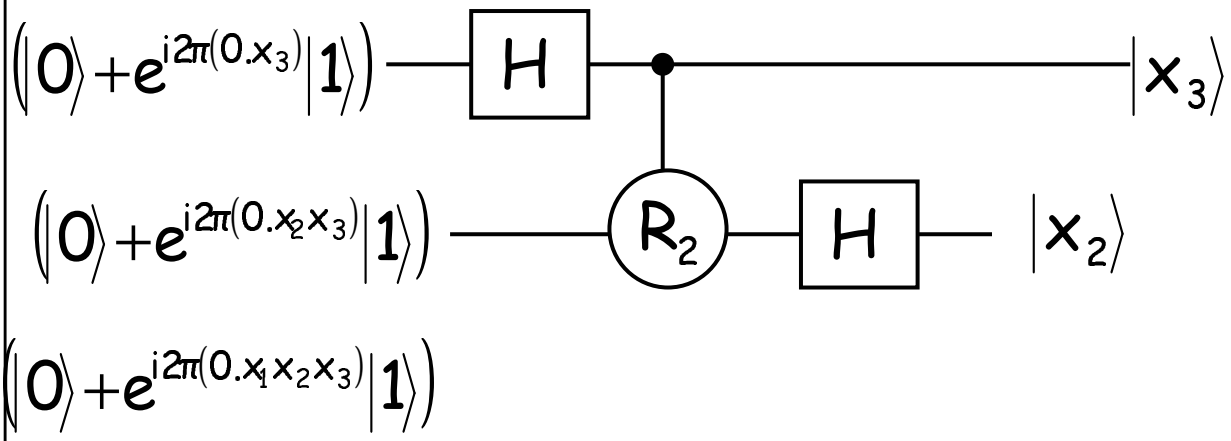
$\left(|0\rangle + e^{i2\pi(0.x_2x_3)}|1\rangle\right)$

$\left(|0\rangle + e^{i2\pi(0.x_1x_2x_3)}|1\rangle\right)$

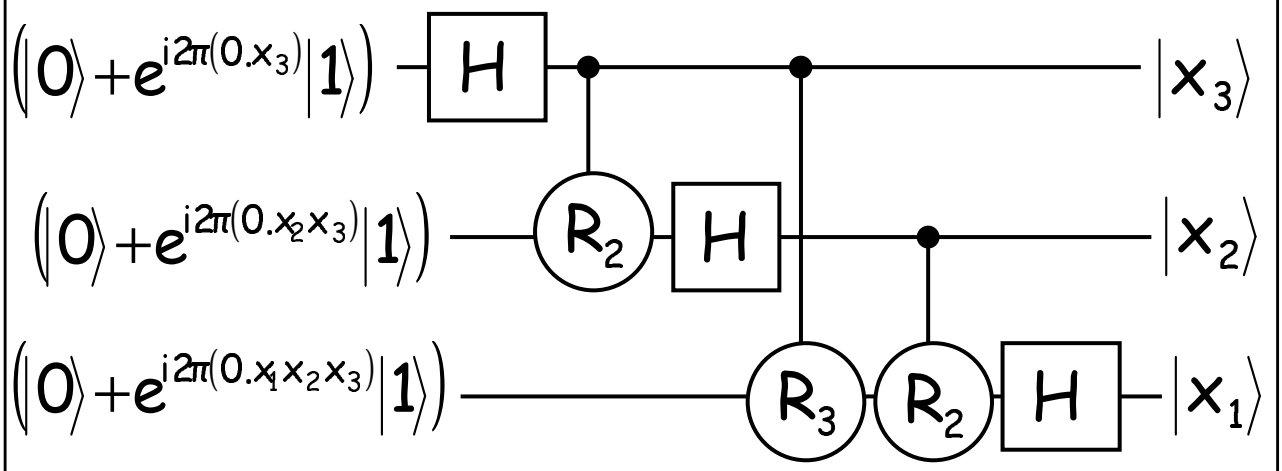
Phase Estimation



Natural Phase Estimation

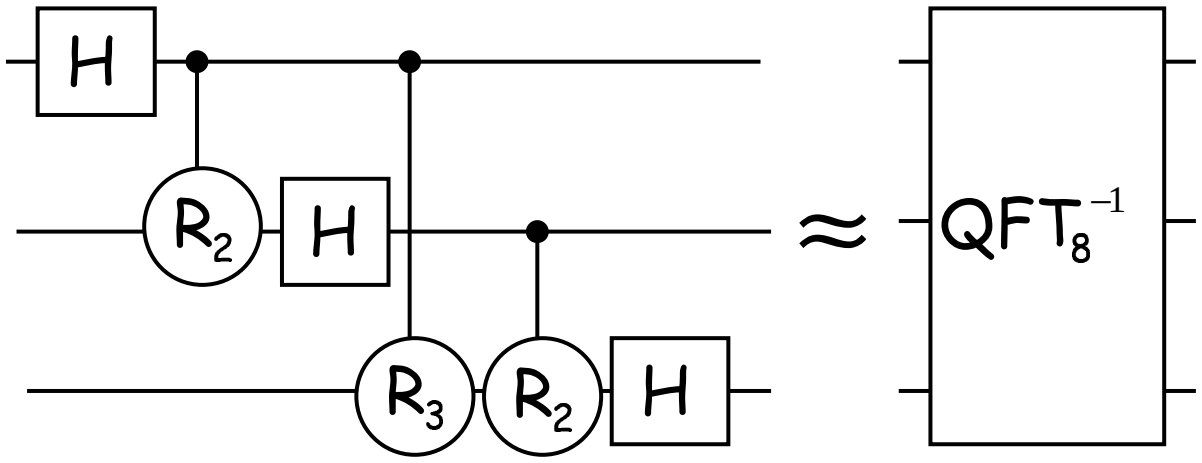


Phase Estimation



Inverse Quantum Fourier Transform

If we reorder the final qubits, we have



What is a (Q)FT?

$$FT_{2^n}: C^{2^n} \to C^{2^n}$$

$$e_j = (0,0,\cdots,1,\cdots,0,0)$$

$$\mapsto \frac{1}{\sqrt{2^n}}\left(1, e^{i2\pi\frac{j}{2^n}}, e^{i2\pi\left(2\frac{j}{2^n}\right)}, \ldots, e^{i2\pi\left((2^n-1)\frac{j}{2^n}\right)}\right)$$

$$FT_{2^n}^{-1}: \frac{1}{\sqrt{2^n}}\left(1, e^{i2\pi\frac{j}{2^n}}, e^{i2\pi\left(2\frac{j}{2^n}\right)}, \ldots, e^{i2\pi\left((2^n-1)\frac{j}{2^n}\right)}\right) \mapsto e_j$$

What is a (Q)FT?

$$FT_{2^n}^{-1}: \frac{1}{\sqrt{2^n}}\left(1, e^{i\varphi}, e^{i2\varphi}, \ldots, e^{i\left((2^n-1)\varphi\right)}\right)$$

$$\mapsto (a_0, a_1, \cdots, a_{2^n-1})$$

$$\left|a_j\right| = \frac{\sin\left(2^n\left(\frac{\varphi}{2\pi} - \frac{j}{2^n}\right)\pi\right)}{2^n \sin\left(\left(\frac{\varphi}{2\pi} - \frac{j}{2^n}\right)\pi\right)}$$

What is a (Q)FT?

$$QFT_{2^n} : H_{2^n} \to H_{2^n}$$

$$|j\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n} e^{i2\pi\left(x\frac{j}{2^n}\right)} |x\rangle$$

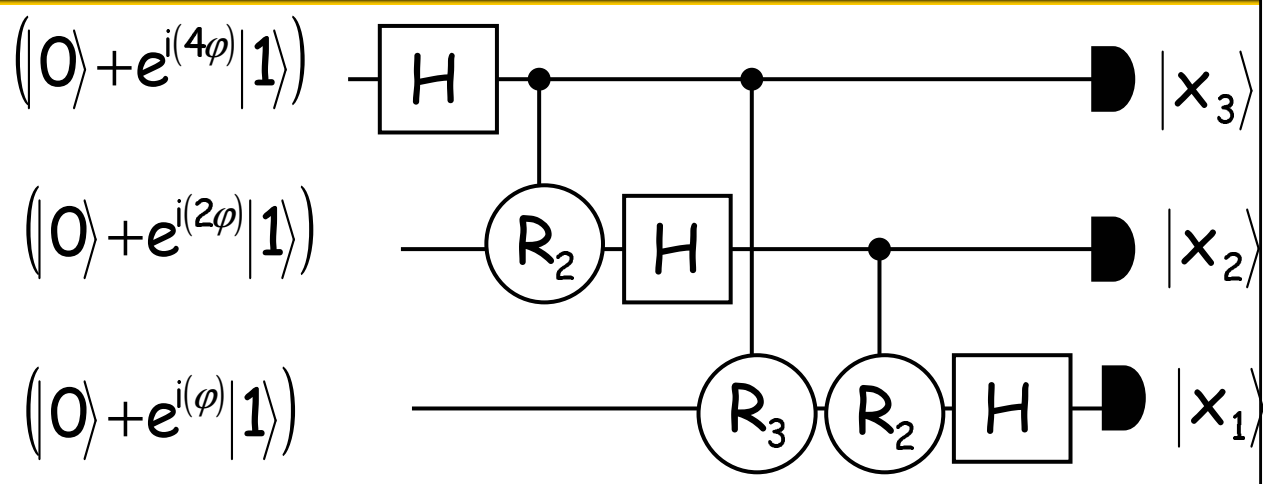
$$QFT_{2^n}^{-1} : \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n} e^{i2\pi\left(x\frac{j}{2^n}\right)} |x\rangle \mapsto |j\rangle$$

What is a (Q)FT?

$$\mathrm{QFT}_{2^n}^{-1}: \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n} e^{ix\varphi} |x\rangle \mapsto \sum_j a_j |j\rangle$$

$$|a_j| = \frac{\sin\left(2^n\left(\frac{\varphi}{2\pi} - \frac{j}{2^n}\right)\pi\right)}{2^n \sin\left(\left(\frac{\varphi}{2\pi} - \frac{j}{2^n}\right)\pi\right)}$$

Phase Estimation: Arbitrary φ

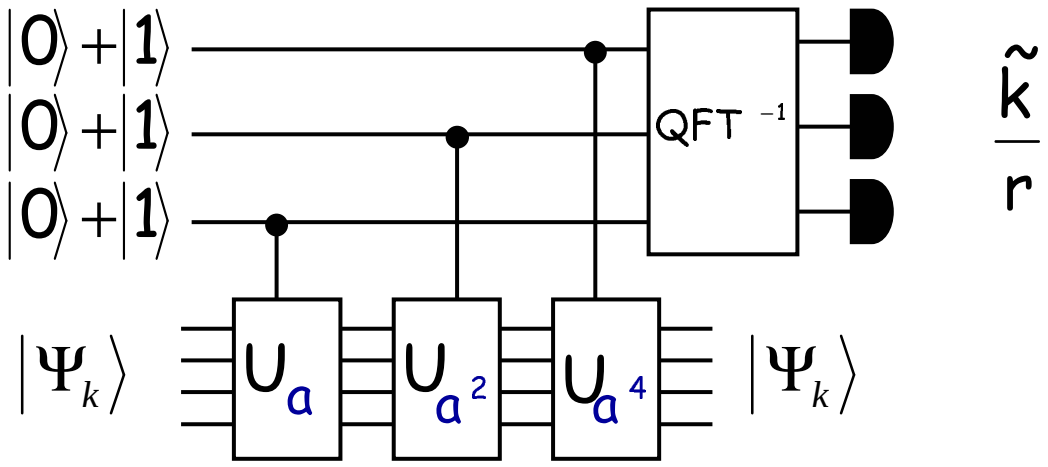


$$\tilde{\varphi} = 2\pi \frac{(4x_1 + 2x_2 + x_3)}{8}$$

$$P\left(\left|\frac{\tilde{\varphi}}{2\pi} - \frac{\varphi}{2\pi}\right| \le \frac{1}{8}\right) \ge \frac{8}{\pi^2}$$

Quantum Factoring

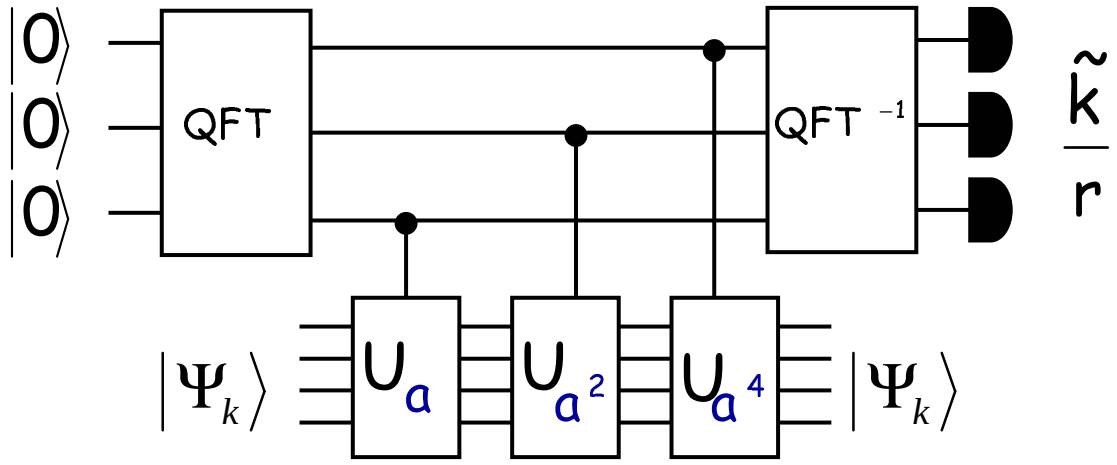
We can efficiently estimate $\frac{k}{r}$



[Kitaev95, CEMM98]

Factoring Network

We are effectively studying the behaviour of the controlled-U_a in a 'very quantum' basis.



Factoring Network

The given network maps

$$|000\rangle|\psi_k\rangle \mapsto \left|\frac{\tilde{k}}{r}\right\rangle|\psi_k\rangle$$

And therefore

$$|000\rangle|1\rangle = \frac{1}{\sqrt{r}}\sum_k |000\rangle|\psi_k\rangle \mapsto \frac{1}{\sqrt{r}}\sum_k \left|\frac{\tilde{k}}{r}\right\rangle|\psi_k\rangle$$

Partial measurements

$$\sum_k \frac{1}{\sqrt{r}} \left| \frac{\tilde{k}}{r} \right\rangle |\psi_k\rangle$$

What do we get when we measure the first register?

In general, we can rewrite

$$\sum_{xy} a_{xy} |x\rangle |y\rangle = \sum_x |x\rangle \left(\sum_y a_{xy} |y\rangle \right)$$

$$= \sum_x b_x |x\rangle |\Phi_x\rangle$$

$$|\Phi_x\rangle = \sum_y \frac{a_{xy}}{b_x} |y\rangle \qquad b_x = \sqrt{\sum_y |a_{xy}|^2}$$

Partial measurements

The probability of measuring x in the first register of $\sum_x b_x |x\rangle|\Phi_x\rangle$ is b_x^2

Partial measurements

Alternatively, we can rewrite

$$\sum_{xy} a_{xy} |x\rangle|y\rangle = \sum_{y} \left(\sum_{x} a_{xy} |x\rangle \right) |y\rangle$$

$$= \sum_{y} c_y \left|\Phi'_y\right\rangle |y\rangle$$

$$\left|\Phi'_y\right\rangle = \sum_{x} \frac{a_{xy}}{c_y} |x\rangle \qquad c_y = \sqrt{\sum_{x} \left|a_{xy}\right|^2}$$

Partial measurements

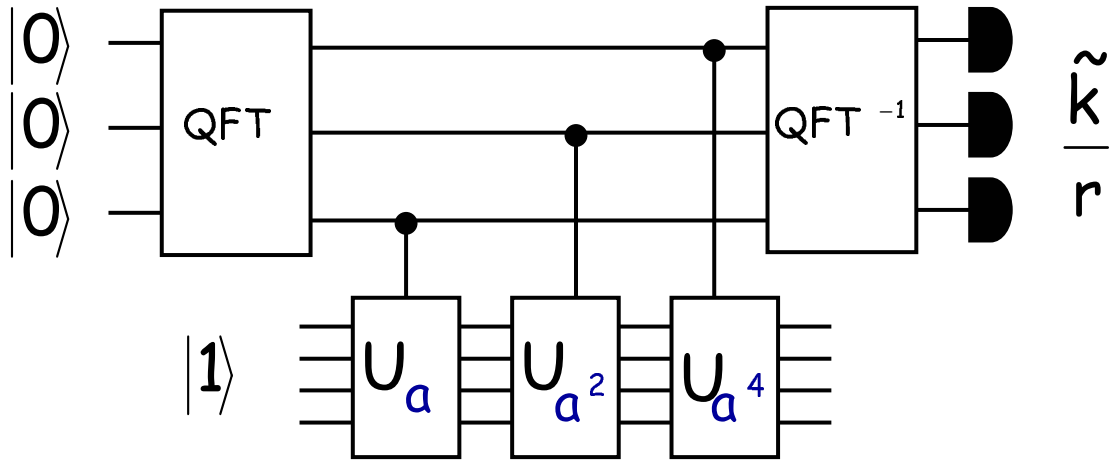
Measuring the first register of $\sum_y c_y \left|\Phi'_y\right\rangle \left|y\right\rangle$ is equivalent to performing a measurement on the state $\left|\Phi'_y\right\rangle$ with probability $c_y^{\,2}$

Partial measurements

Measuring the first register of $\sum_k \frac{1}{\sqrt{r}} \left| \frac{\tilde{k}}{r} \right\rangle |\psi_k\rangle$ is equivalent to performing a measurement on the state $\left| \frac{\tilde{k}}{r} \right\rangle$ with probability $\frac{1}{r}$

Factoring Network

We are effectively studying the behaviour of the controlled-U_a in a 'very quantum' basis.



[CEMM98] show this is equivalent to [Shor94]

Complexity comparison

The best rigorous classical algorithms use

$e^{O(\sqrt{\log(N)\log\log(N)})}$ operations

The best heuristic classical algorithms use

$e^{O((\log(N)^{\frac{1}{3}}\log\log(N)^{\frac{2}{3}})}$ operations

The quantum algorithm uses poly(log(N))

$= e^{O(\log\log(N))}$ operations

Hidden Subgroup

This approach allows us to solve efficiently any "Abelian Hidden Subgroup Problem" (see [ME98],[M99],[NC00])

$$f : G \rightarrow X$$

$$K \leq G$$

$$f(x) = f(y) \quad \Leftrightarrow \quad x - y \in K$$

Find K

Hidden Affine Functions

Hidden Affine Functions:

$$f : Z_p^n \rightarrow Z_p^m$$

$$x \rightarrow Mx + b$$

Find M using only m evaluations of f
(instead of n+1) (D,BV,CEMM,H,M)