Quantum Error Correction II

Robust Quantum Information Processing

Manny

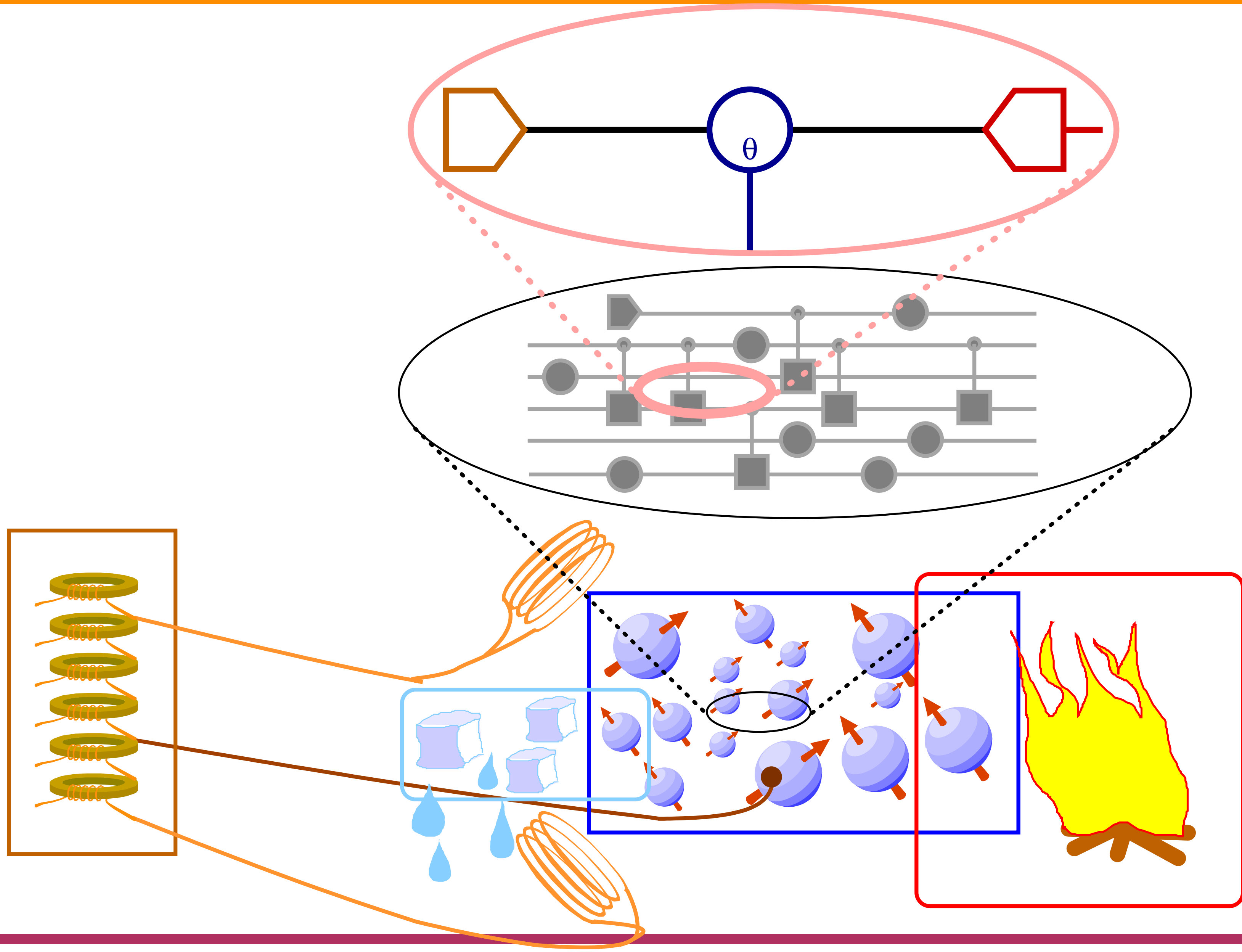
- Threshold theorems.
- Requirements for scalability.
- Physical noise models.
- Methods for scalability.

References

- Prehistory:
  - Quantum Zeno effect: Misra&Sudarshan 1977 [18].
  - Deutsch 1993, Barenco&*al.* 1996 [3].
- Discovery and Theory:
  - Shor 1995 [21], Steane 1995 [23].
  - Bennett&DiVincenzo&Smolin&Wootters 1996 [4], Knill&Laflamme 1996 [12].
  - Calderbank&Shor 1996 [6], Gottesman 1996 [8], Calderbank&Rains&Shor&Sloane 1997 [5].
- Fault tolerance and threshold accuracies:
  - Shor 1996 [22], Kitaev 1997 [11].
  - Aharonov&Ben-Or 1996 [1, 2], Knill&Laflamme&Zurek 1996 [15], Gottesman&Preskill 1997 [9, 20], Dür&Briegel&*al.* 1999 [7].
  - Gottesman 1997 [9], Gottesman&Chuang 1999 [10]
- Toward subsystems:
  - Quasi-particles . . .
  - Zanardi&Rasetti 1997 [27], Lidar&Chuang&Whaley 1998 [17].
  - Viola&Knill&Lloyd 1998 [25, 24, 26].
  - Knill&Laflamme 1996 [12], Knill&Laflamme&Viola 2000 [14].

General reference: (M)ike, Ch. 10. Nielsen&Chuang 2001 [19]

QIP System Overview



The Threshold Theorems

- The Accuracy Threshold Theorem:
  Assume the *requirements for scalable computing*. If the error per gate (including "no-op") is less than a threshold, then it is possible to efficiently quantum compute arbitrarily accurately.

  - Error Thresholds:
    - Worst case $> 10^{-6}$.
    - Estimate $> 10^{-4}$.
    - Communication $> 10^{-2}$.
    - Erasure $> 10^{-2}$.
    - Z-measurement $\geq .5$.

Shor 1996[22], Kitaev 1996[11], Aharonov&Ben-Or 1996[1], Knill& *al.* 1996[16].

Sufficient Requirements for Scalability

Realize QIP task with M gates and $\mathbf{noops}$, N qubits.

- Systems:
  - $\mathrm{polylog}(M)N$ independent ≥ 2 state subsystems.
- Preparation:
  - Can reset subsystem to $|0\rangle$ anytime.
- Control:
  - Universal two-system gates can be applied.
- Parallelism:
  - Can apply gates in parallel.
  - Or: Perfect long-term memory.
- Noise:
  - Sufficiently weak.
  - Quasi-independent.

Types of Noise

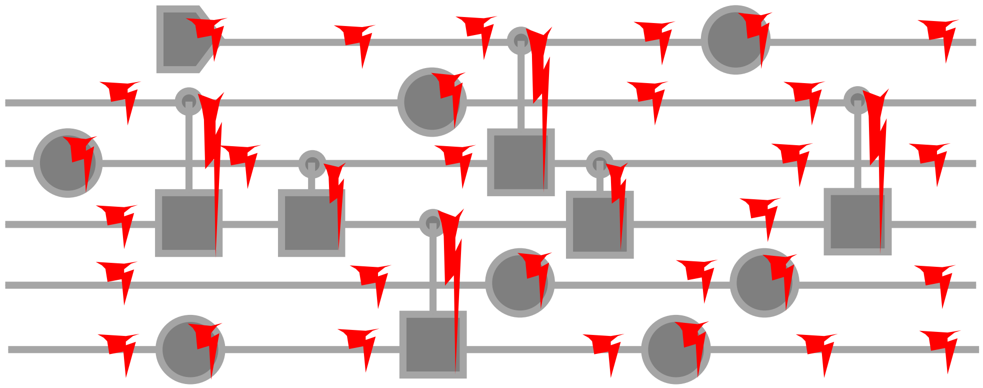
- Quantum noise in QIP:
  Any unwanted effect in quantum systems.

- By temporal behavior.
  - Step-wise independent, discrete.
  - Markovian.
  - Relaxation: Decay to equilibrium.
    - Depolarization.
    - Dissipation, thermal relaxation.
    - Decoherence: Loss of phase.
  - Stationary.

- By spatial behavior.
  - Local or factorizable.
  - Symmetric.
  - Linear.

- By origin.
  - Thermal.
  - Control errors, faults.
    - Miscalibration.
    - Over/under-rotation.
  - Stray fields or inhomogeneity.

Noise Analysis



- Error locations:

  For each gate U_i including $U_i = \mathbf{noop}$:

$$U_{i\text{actual}} = E_i U_{i\text{intended}}$$

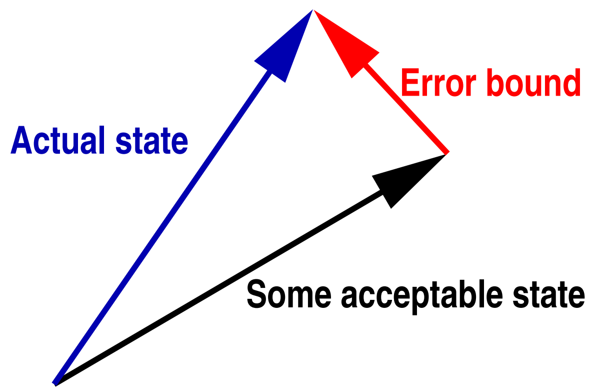
Locality: E_i acts on the same qubits as U_i.

- Error expansion with "environment":

$$|\psi\rangle \rightarrow \sum_e |e\rangle_{\mathrm{E}}\ E_{e,n}U_n \cdots E_{e,2}U_2\ E_{e,1}U_1|\psi\rangle$$

The $|e\rangle_{\mathrm{E}}$ need not be orthogonal.

A Measure of Noise



- Wavefunction:

r.h.s. states need not be normalized

$$|\text{output}\rangle = |\text{acceptable}\rangle + |\text{error}\rangle$$

  - Error probability is bounded by $\langle\text{error}|\text{error}\rangle$.

- Density operator:

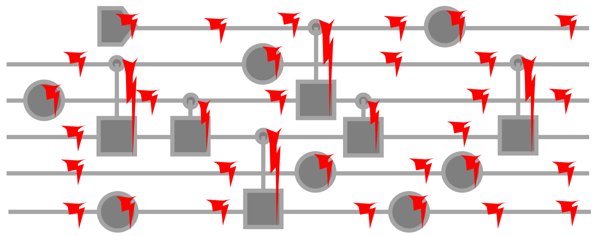
Error operator need not be a state.

$$\rho_{\text{out}} = \rho_{\text{acceptable}} + \rho_{\text{error}}$$

  - Error probability is bounded by $(\text{tr}|\rho_{\text{error}}| + |\text{tr}\rho_{\text{error}}|)/2$.

- Acceptable states need not be unique.

Quasi-independent Noise



- Noise given by error expansion:

$$|\psi\rangle \to \sum_e |e\rangle_{\mathsf{E}}\, E_{e,n} U_n \cdots E_{e,2} U_2\, E_{e,1} U_1 |\psi\rangle$$

- The *support* of error term e is
  the set of i such that $E_{e,i}$ is not $\mathbb{I}$.
- *Quasi-independent* with probability p if for each $I \subseteq \{1, \ldots, n\}$
  probability of errors with support $\supseteq I$ is $\leq p^{|I|}$.

Threshold: $p < 10^{-6}$ at worst.

Independent Noise Models

- Systematic models.
  - Only one environment state $|e\rangle_{\mathsf{E}}$.

$$\begin{aligned} |\psi\rangle &\rightarrow |e\rangle_{\mathsf{E}}\, V_n U_n \cdots V_2 U_2\, V_1 U_1 |\psi\rangle \\ &= |e\rangle_{\mathsf{E}}\, (\mathbb{I} + E_n) U_n \cdots (\mathbb{I} + E_2) U_2\, (\mathbb{I} + E_1) U_1 |\psi\rangle \end{aligned}$$

  - Examples: Frequency errors, miscalibration.

Correctable by classical control.

- Random models.
  - Orthogonal environments at each site.

$$\begin{aligned} |\psi\rangle \rightarrow\ & (|0\rangle_{\mathsf{E_n}} \mathbb{I} + \ldots + |k_n\rangle_{\mathsf{E_n}} E_{n,k_n}) U_n \cdots \\ & (|0\rangle_{\mathsf{E_2}} \mathbb{I} + \ldots + |k_2\rangle_{\mathsf{E_2}} E_{2,k_2}) U_2 \\ & (|0\rangle_{\mathsf{E_1}} \mathbb{I} + \ldots + |k_1\rangle_{\mathsf{E_1}} E_{1,k_1}) U_1 |\psi\rangle \end{aligned}$$

  - Example: Independent thermal relaxation.

Simple Random Models

- Random bit flips $\mathcal{X}(p)$.
  - One qubit: $\sqrt{1-p}|0\rangle_{\mathsf{E}}\mathbb{I} + \sqrt{p}|1\rangle_{\mathsf{E}}\sigma_x$.
  - Two qubits?

Threshold?

- Depolarizing noise $\mathcal{P}(p)$.
  - One qubit:
    $\sqrt{1-\frac{3}{4}p}\,|0\rangle_{\mathsf{E}}\mathbb{I} + \sqrt{\frac{1}{4}p}\,|1\rangle_{\mathsf{E}}\sigma_x + \sqrt{\frac{1}{4}p}\,|2\rangle_{\mathsf{E}}\sigma_y + \sqrt{\frac{1}{4}p}\,|3\rangle_{\mathsf{E}}\sigma_z$.
  - Two qubits?
  - Other independent models can be “twirled” into this.

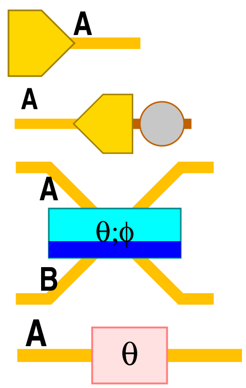
Threshold: $p < 10^{-4}$?

- Erasure/detected loss $\mathcal{L}(p)$.
  - $\sqrt{1-p}|0\rangle_{\mathsf{E}}\mathbb{I} + \sqrt{p}|1\rangle_{\mathsf{E}}\mathcal{P}(1)$.
  - E is accessible.

Threshold: $p < 10^{-2}$ at worst.

eLOQC Designer Errors

- eLOQC: Efficient linear optics quantum computation.



$\Rightarrow$ 1. Non-deterministic QC (nLOQC)

$\Rightarrow$ 2. Near-deterministic QC ($\tilde{\text{d}}$LOQC)

$\boxed{\Rightarrow \text{3. Efficient QC (eLOQC)}}$

- Source of non-determinism: Z-measurement errors.
  - Elementary gates succeed, but...
  - ...with probability f: qubit measured.

Threshold: $f < .5$ at worst.

Knill&Laflamme&Milburn 2001 [13]

Methods for Scalability

- Error-correcting subsystems.

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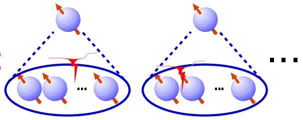
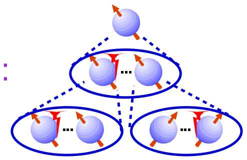
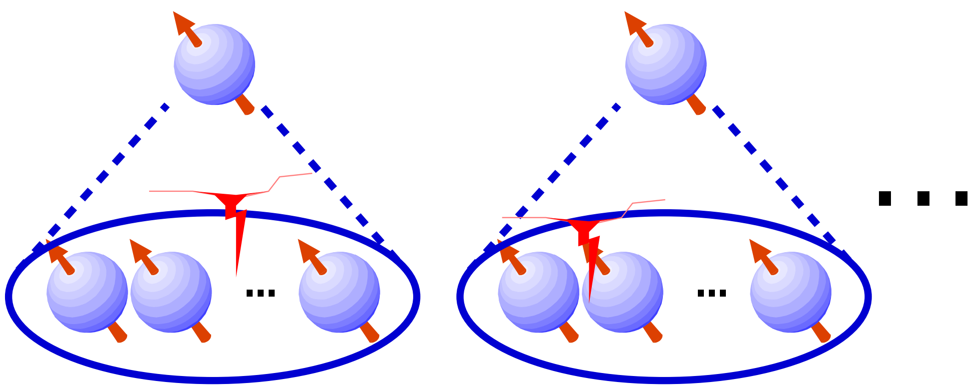


- Concatenation.

Go to:

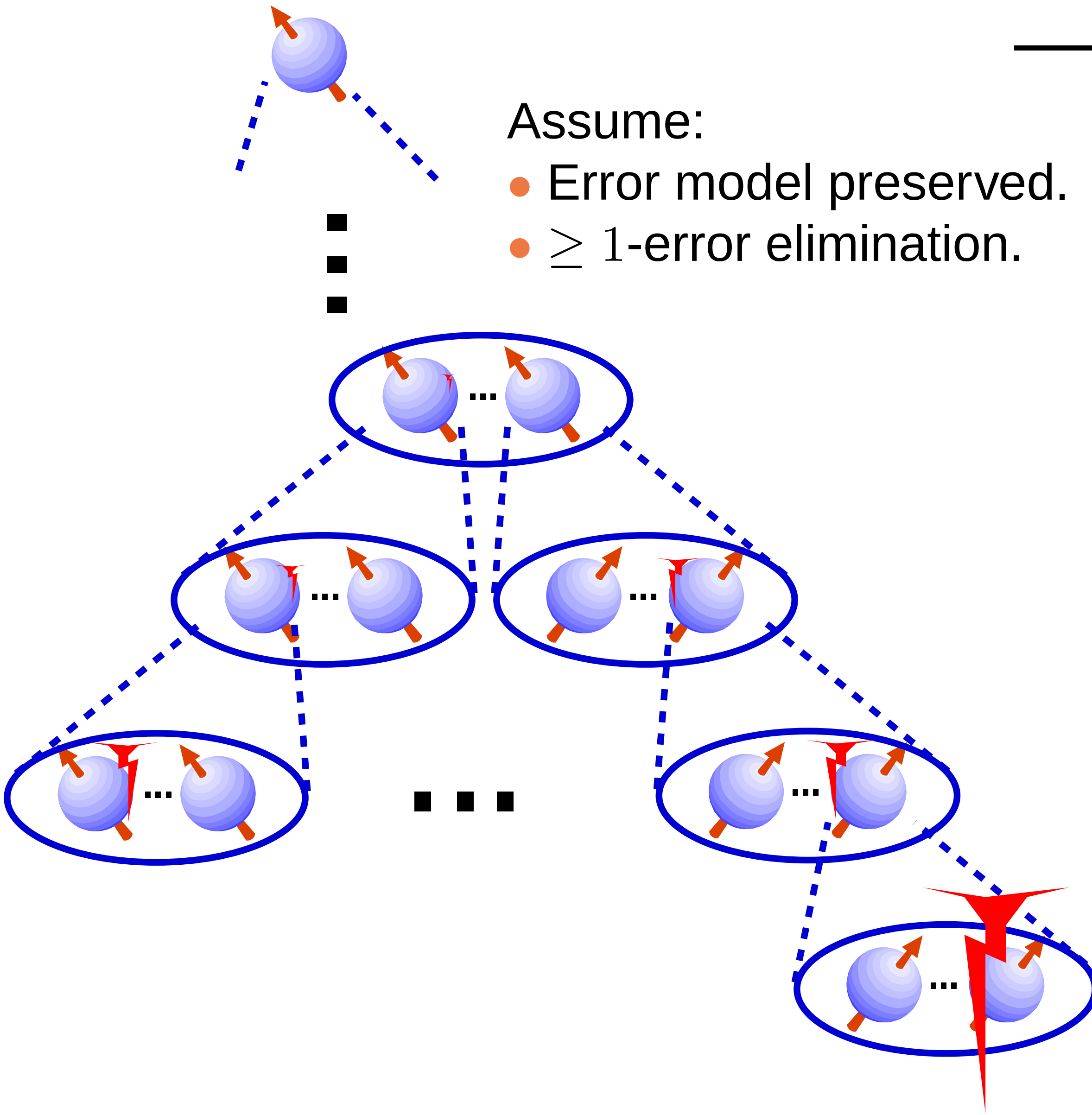


Error-correcting Subsystems



Back to: Methods for Scalability

Concatenation



Assume:

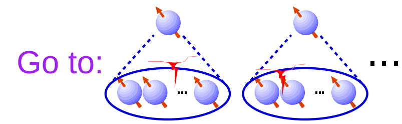
- Error model preserved.
- ≥ 1-error elimination.

Level	Error rate
k	$\leq C^{2^k-1}p^{2^k}$
⋮	
4	$\begin{cases} p_4 \leq C(C^3p^{2^2})^2 \\ \quad = C^{2^3-1}p^{2^3} \end{cases}$
3	$\begin{cases} p_3 \leq C(Cp^2)^2 \\ \quad = C^{2^2-1}p^{2^2} \end{cases}$
2	$p_2 \leq Cp^2$
1	p

Back to: Methods for Scalability

Methods for Scalability

- Error-correcting subsystems.



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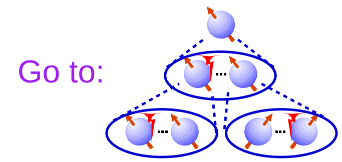
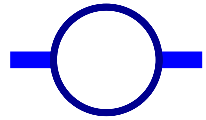


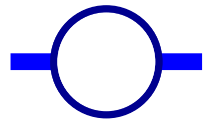
$\Rightarrow$ unbounded distance communication.

Pauli Product Operator Rotations

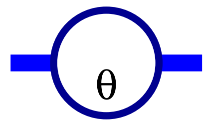
- One qubit rotations and gates:
  - $X_{90^\circ} = e^{-i\sigma_x \pi/4}$



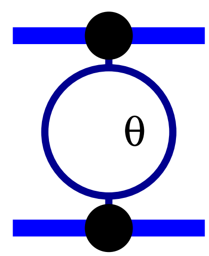
  - $Y_{-90^\circ} = e^{i\sigma_y \pi/4}$



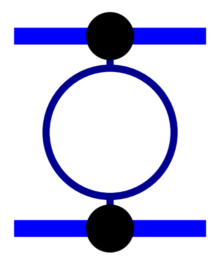
  - $Z_\theta = e^{-i\sigma_z \theta/2}$



- Two qubit rotations and gates:

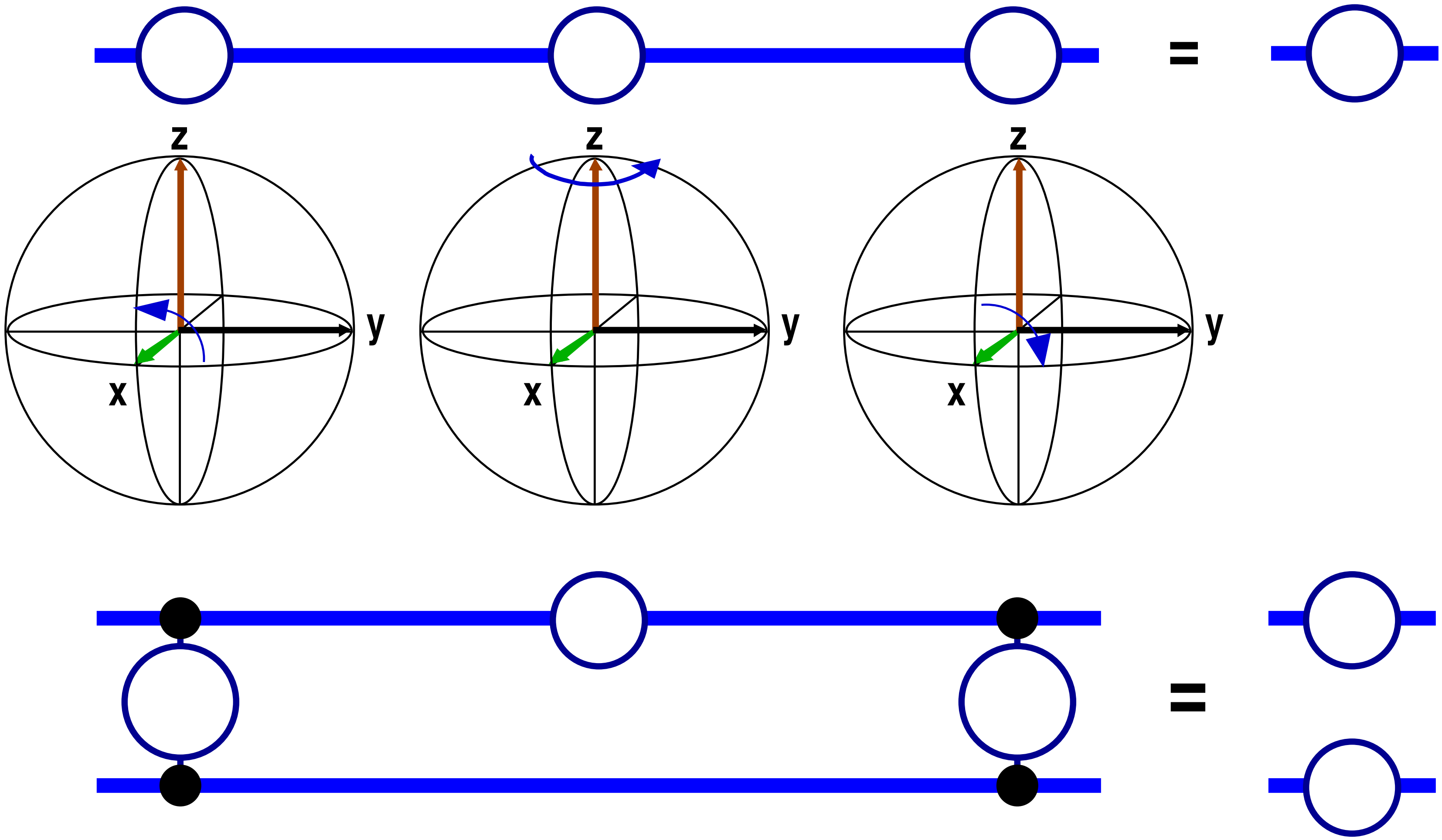


  - $\left(Z^{(1)}Z^{(2)}\right)_\theta = e^{-i{\sigma_z}^{(1)}{\sigma_z}^{(2)}\theta/2}$

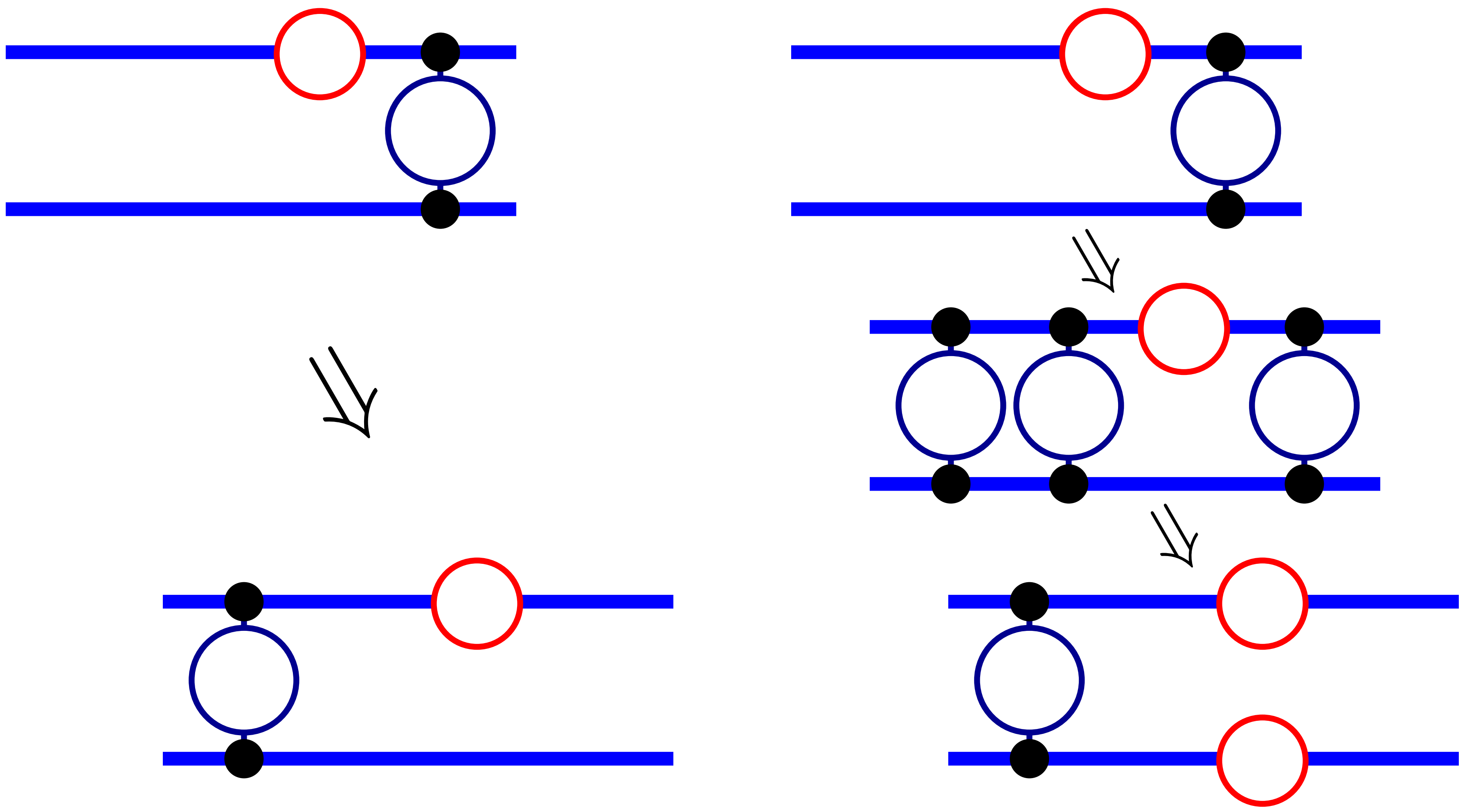


  - $\left(Y^{(1)}Z^{(2)}\right)_{90^\circ} = e^{-i{\sigma_y}^{(1)}{\sigma_z}^{(2)}\pi/2}$

Rotation Equivalences



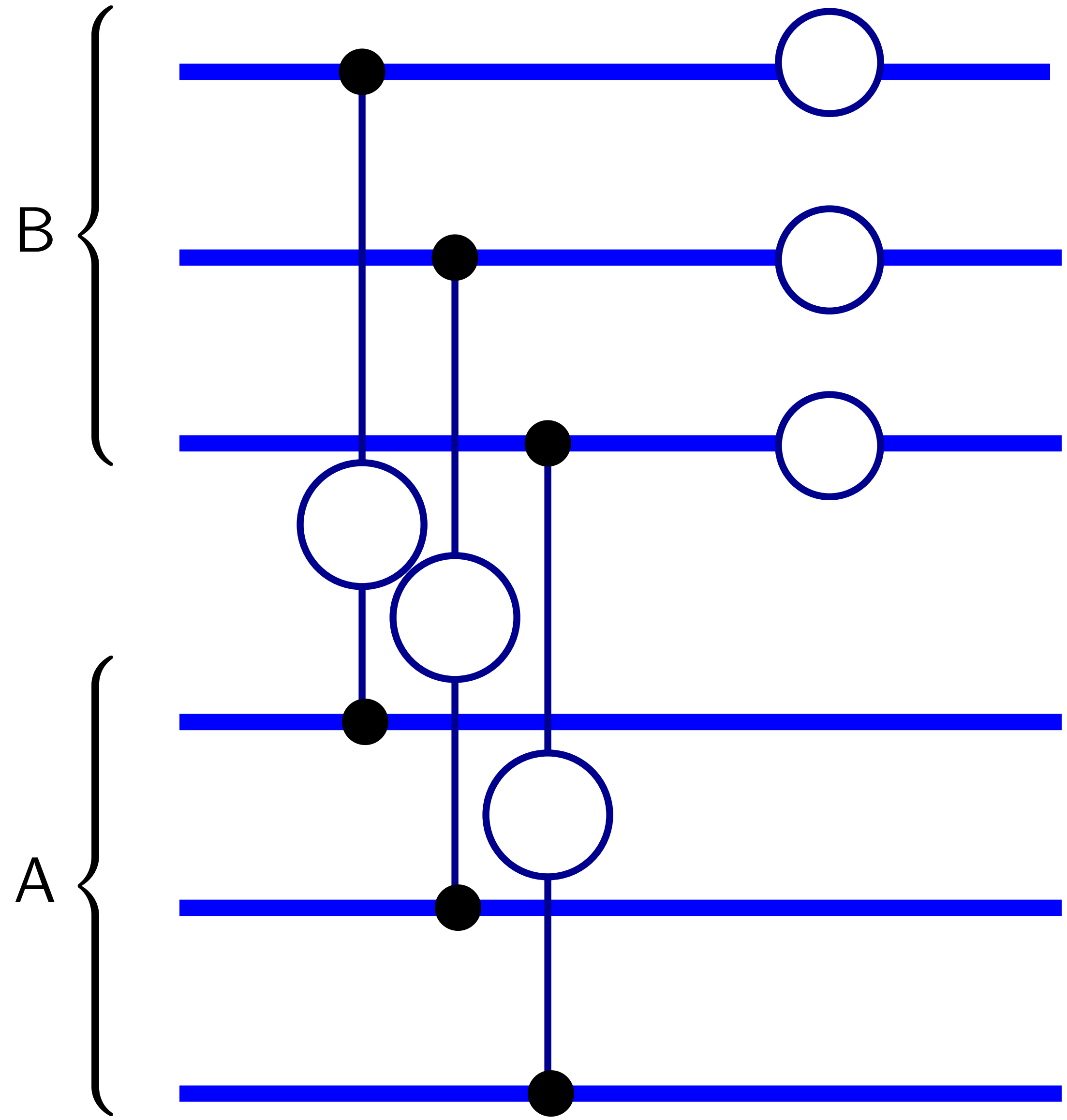
Error Propagation



Transversally Encoding Operations

Repetition code for bit flips.
Stabilizer: $\langle\, ZZI, IZZ \,\rangle$
$|0\rangle_L = |000\rangle$, $|1\rangle_L = |111\rangle$

- “Encoded” $\mathbf{cnot}$ ...



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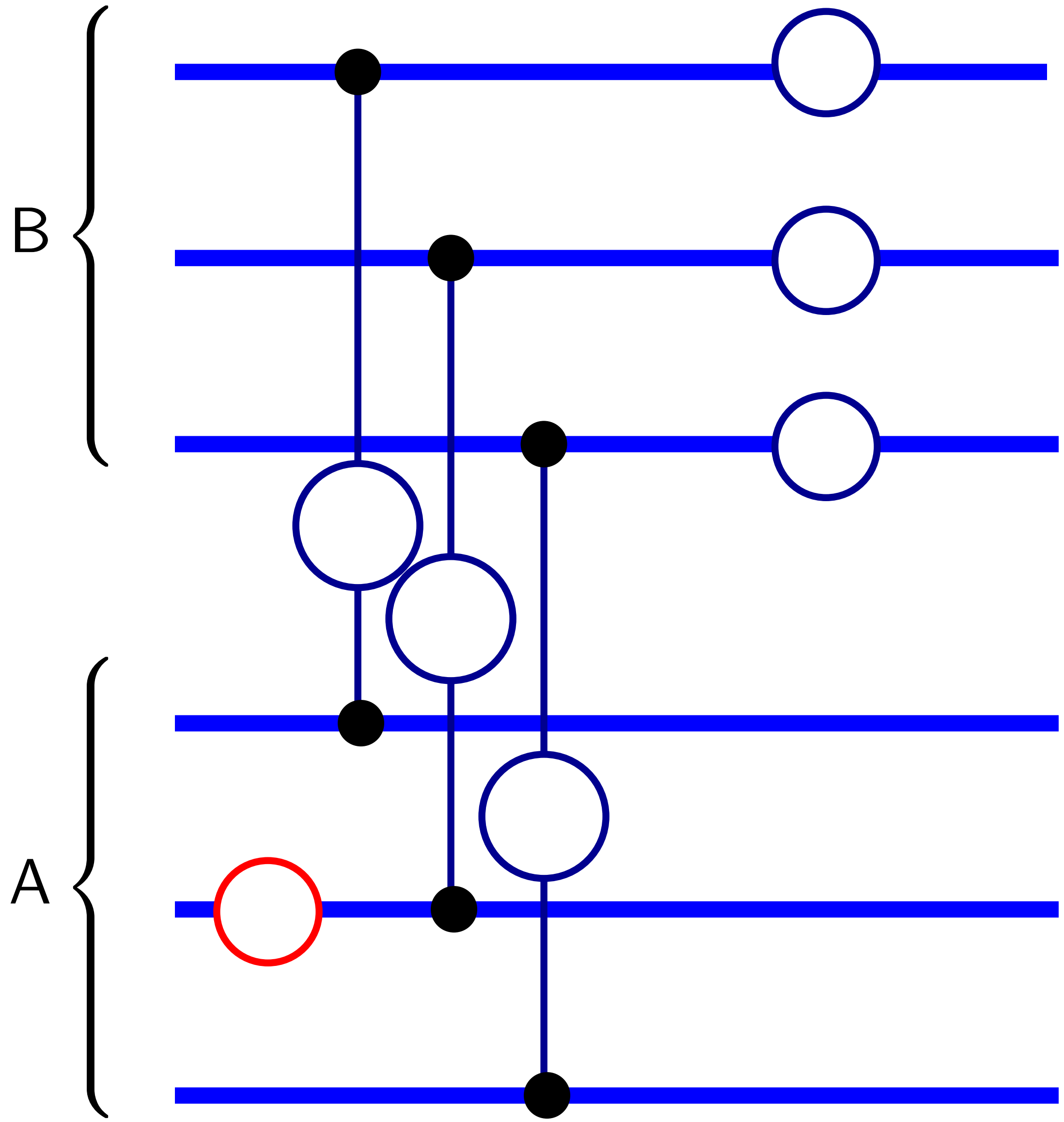

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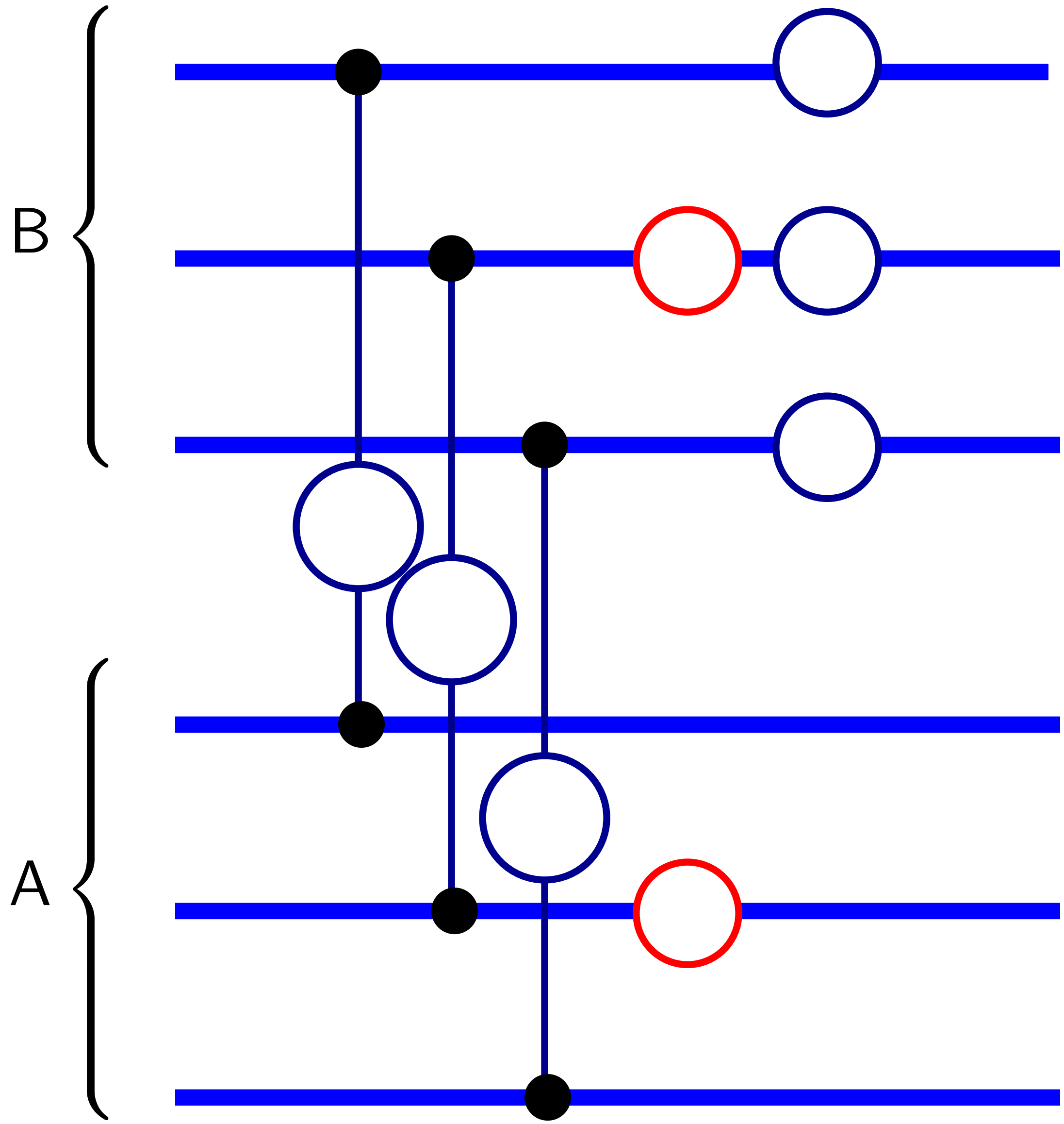

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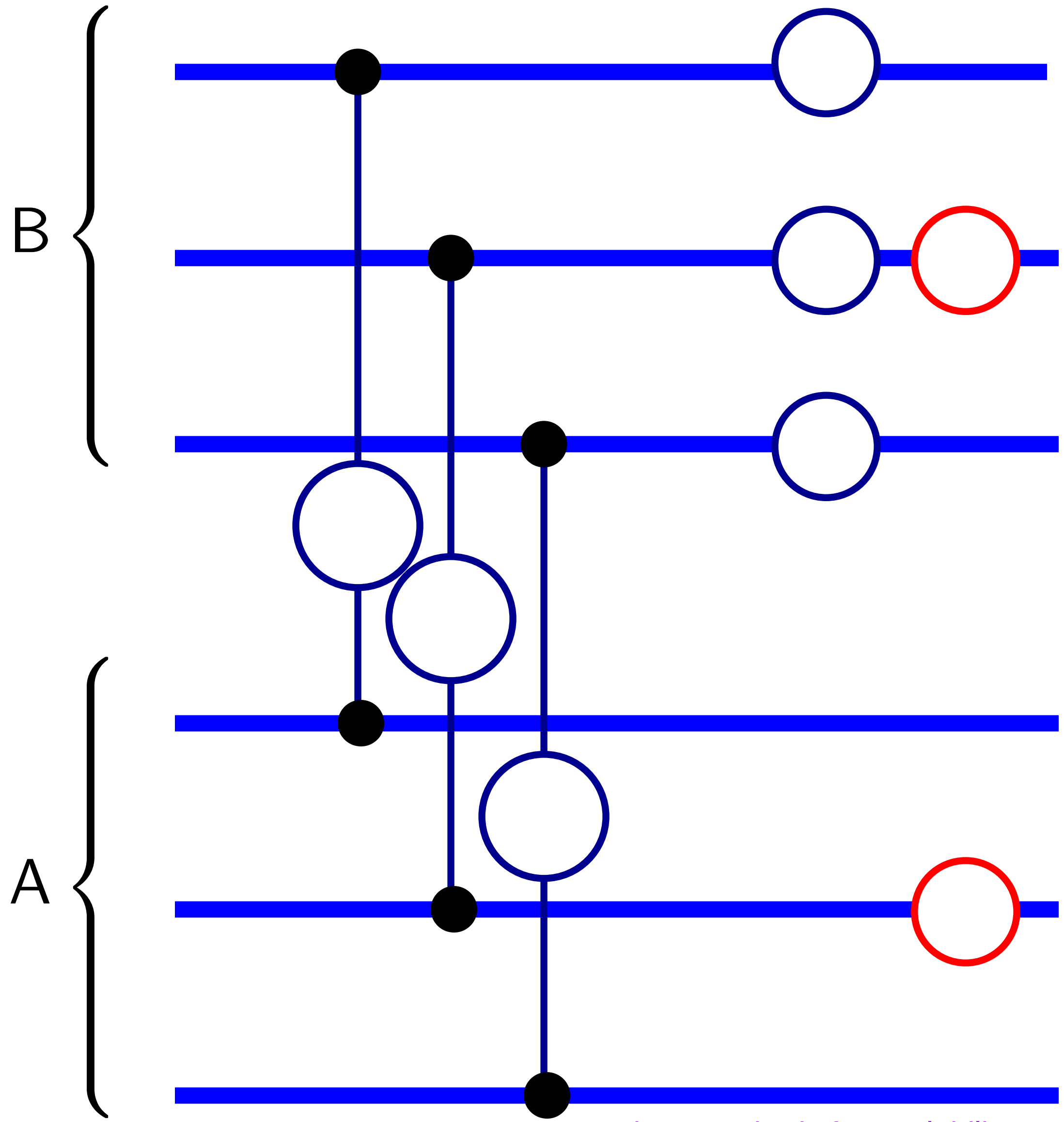
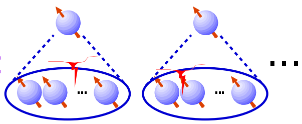
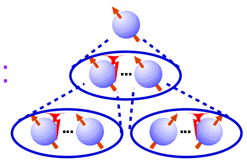
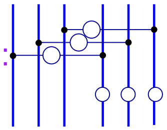


Back to: Methods for Scalability

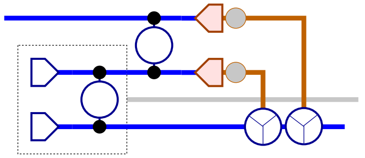
Methods for Scalability

- Error-correcting subsystems. Go to: 
- Concatenation. Go to: 

$\Rightarrow$ unbounded distance communication.

- Transversally encoded operations. Go to: 
- Measurement or Operation

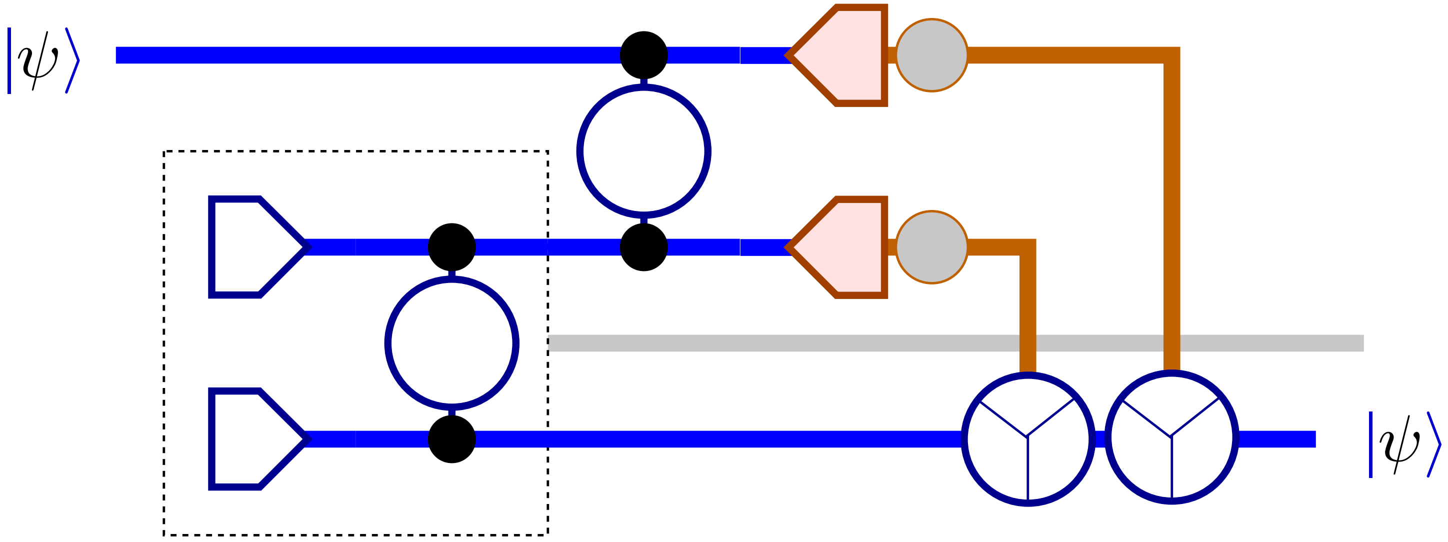
$=$ state preparation $+$ teleportation.

Go to: 

- Robust error detection and recovery.

Jump to: Conclusion

Teleportation

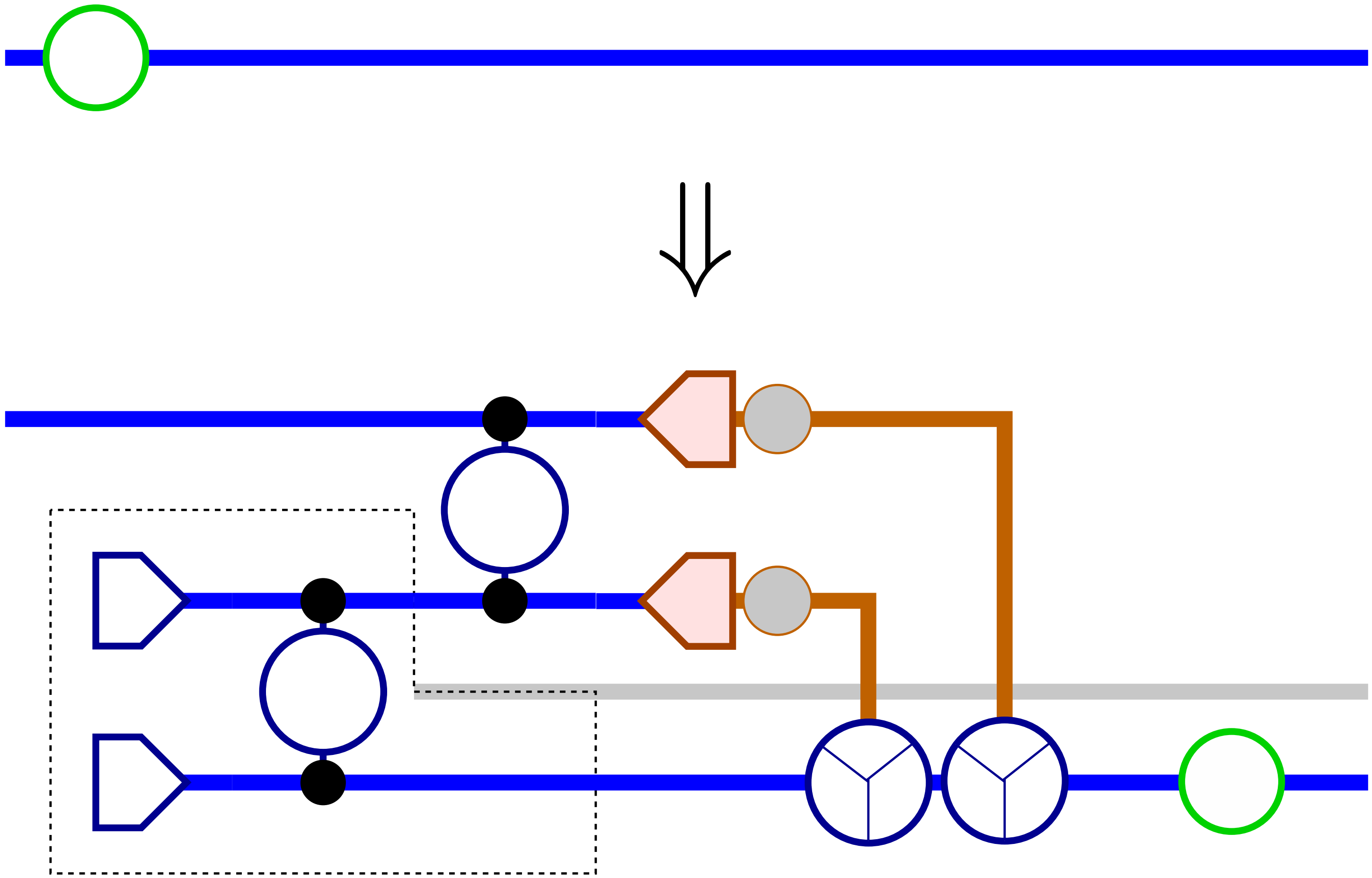


$$\text{a} = +1 \Rightarrow X_{180^\circ} \quad , \quad \text{a} = -1 \Rightarrow \mathbb{I}$$

and

$$\text{b} = +1 \Rightarrow Z_{180^\circ} \quad , \quad \text{b} = -1 \Rightarrow \mathbb{I}$$

Operation = Preparation + Teleportation I



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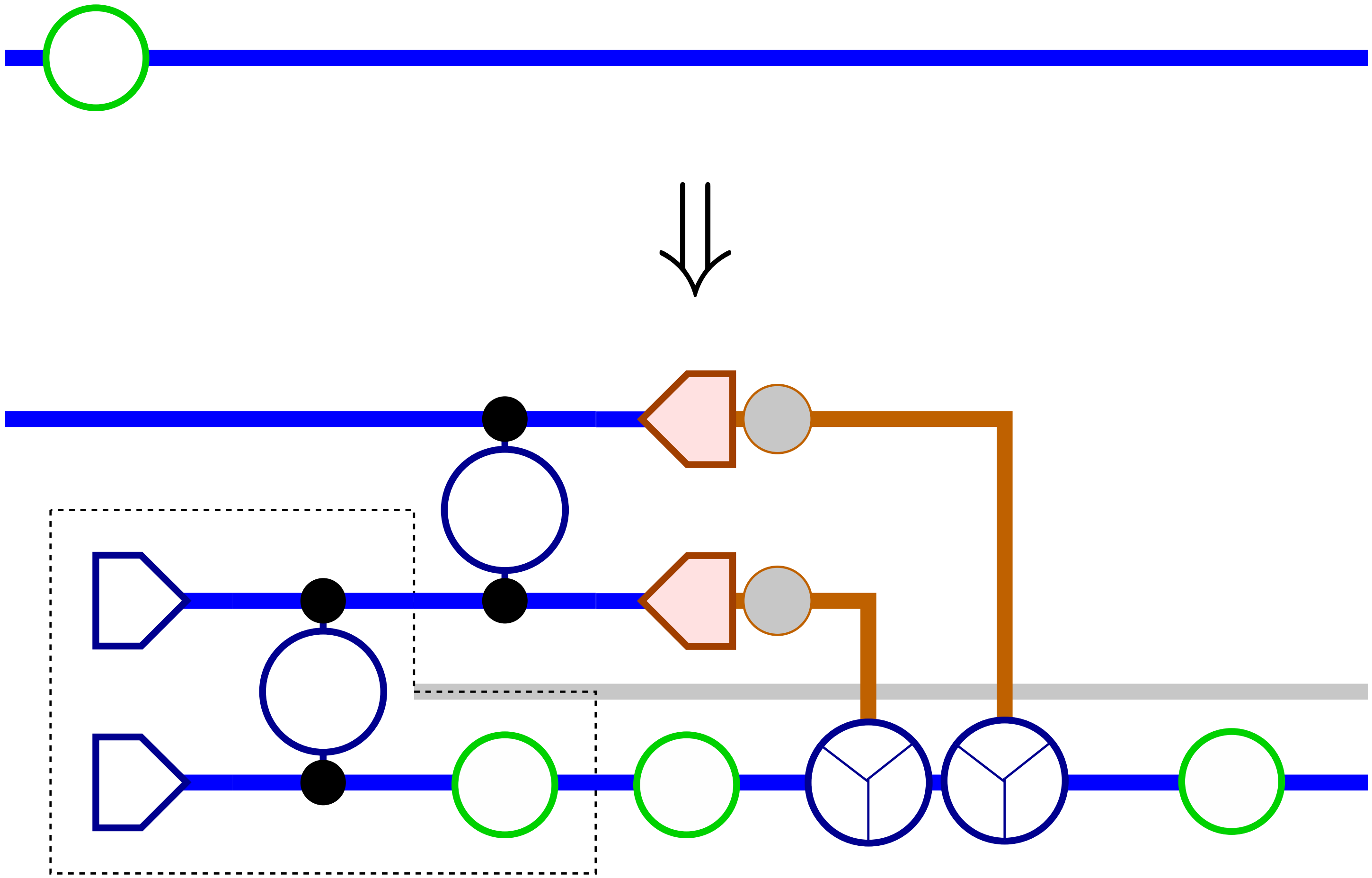

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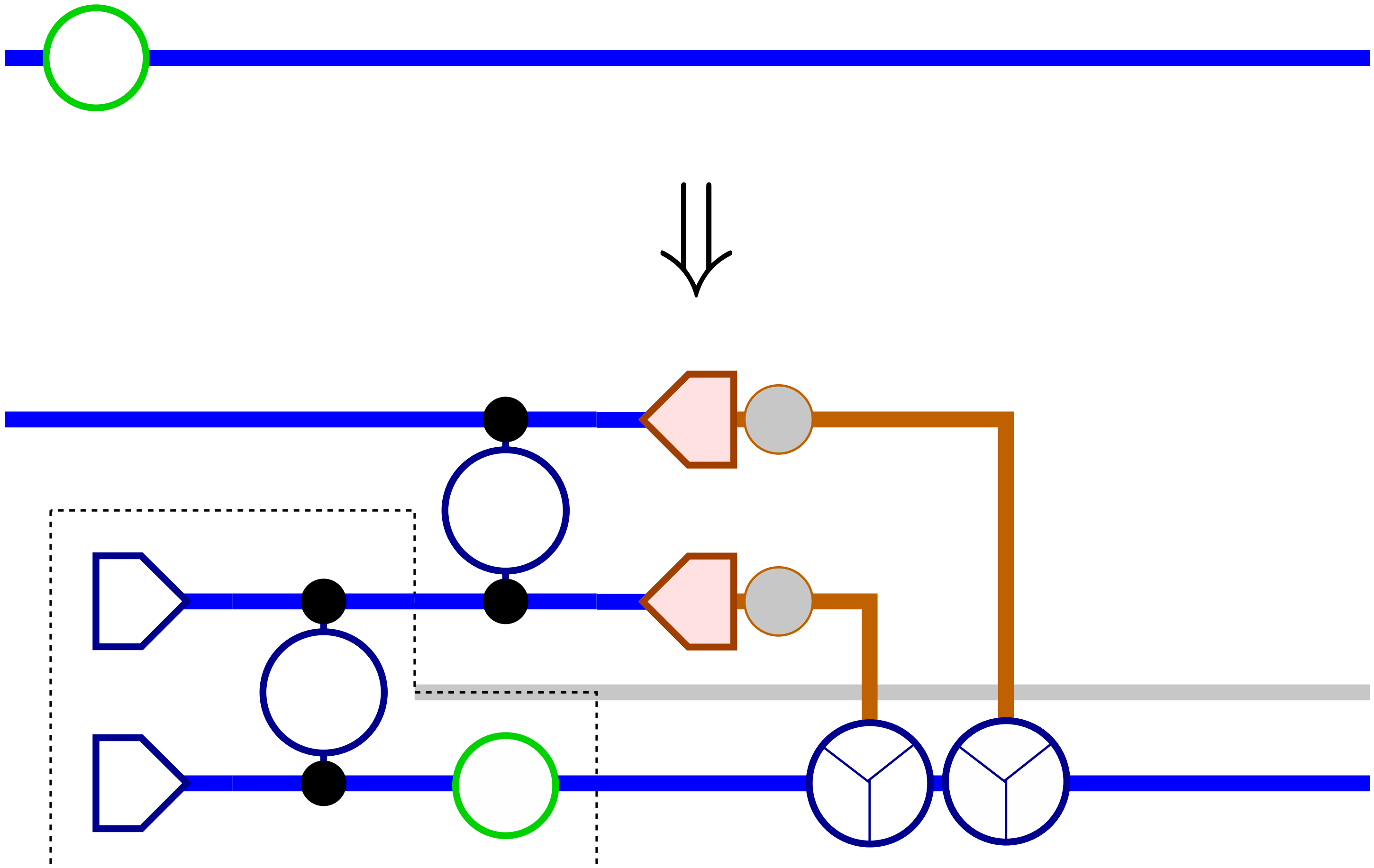
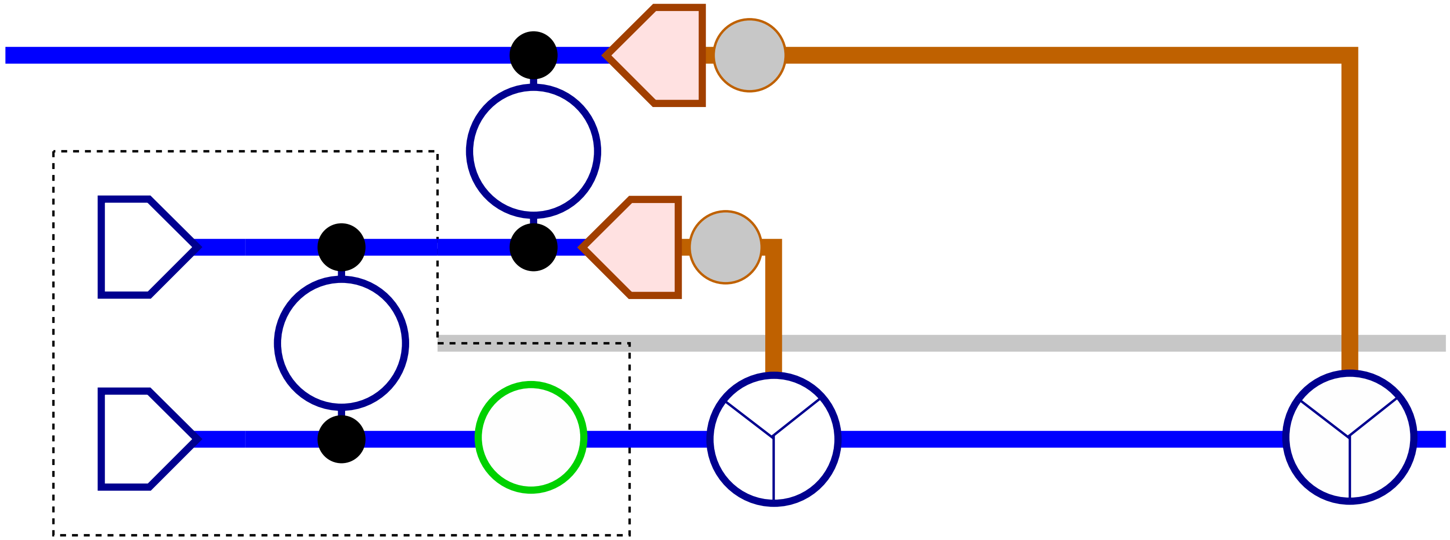


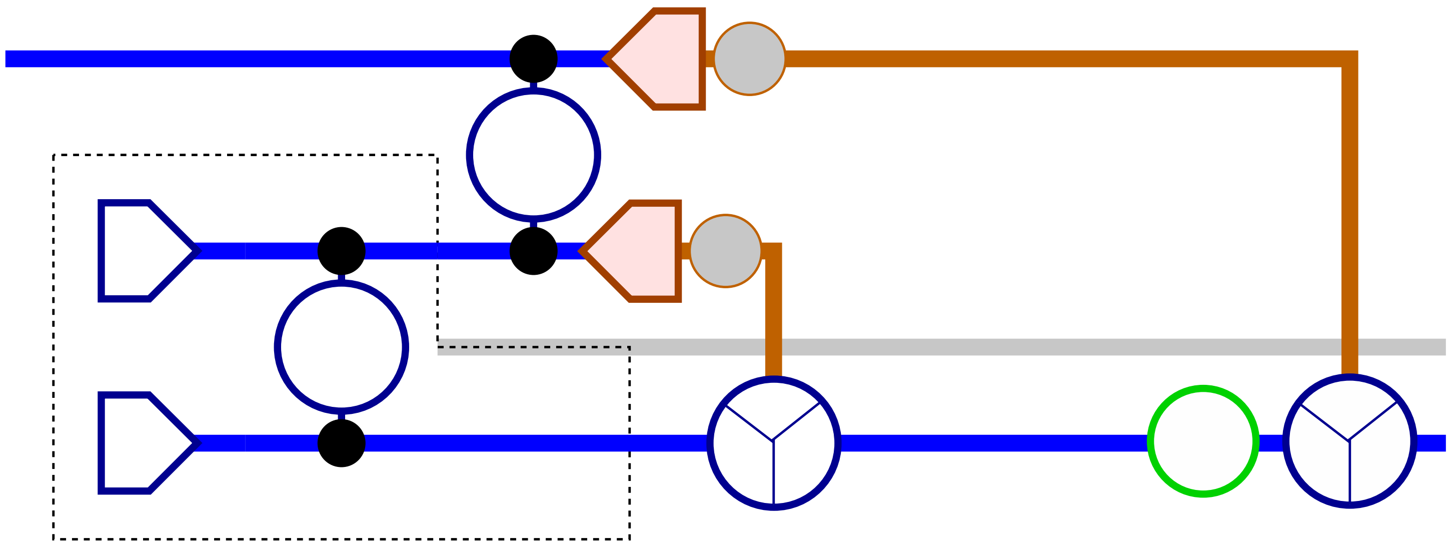
Gottesman&Chuang 1999 [10]

To: Advantages of Teleportation

Operation $=$ Preparation $+$ Teleportation II



Operation $=$ Preparation $+$ Teleportation II



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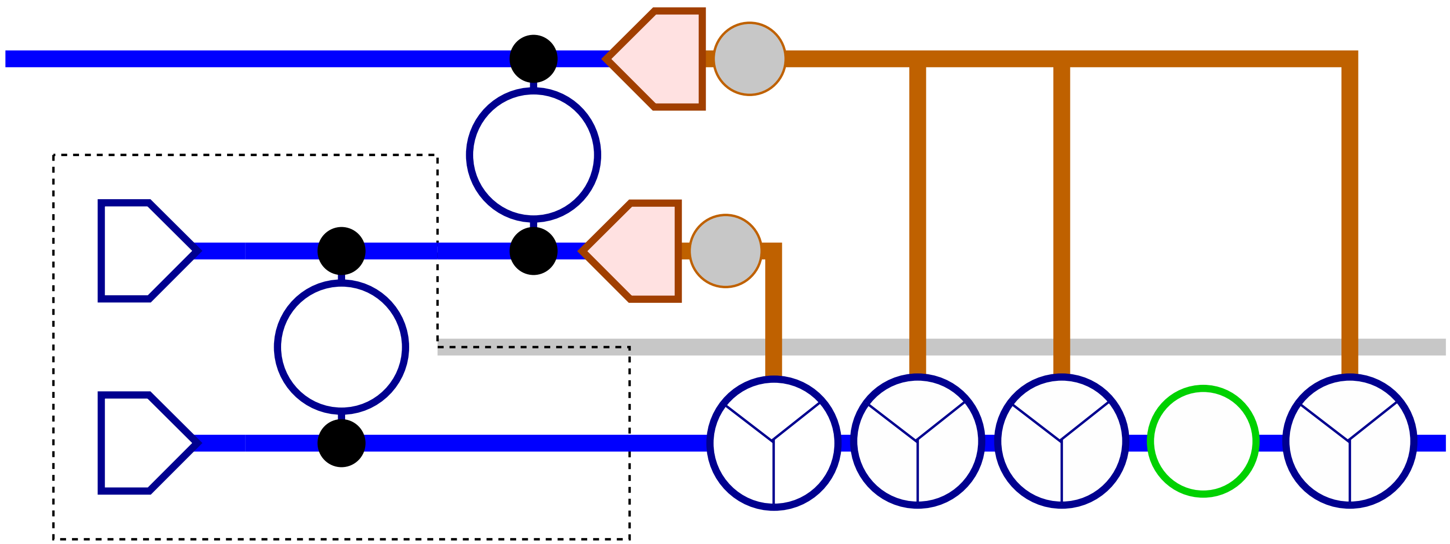

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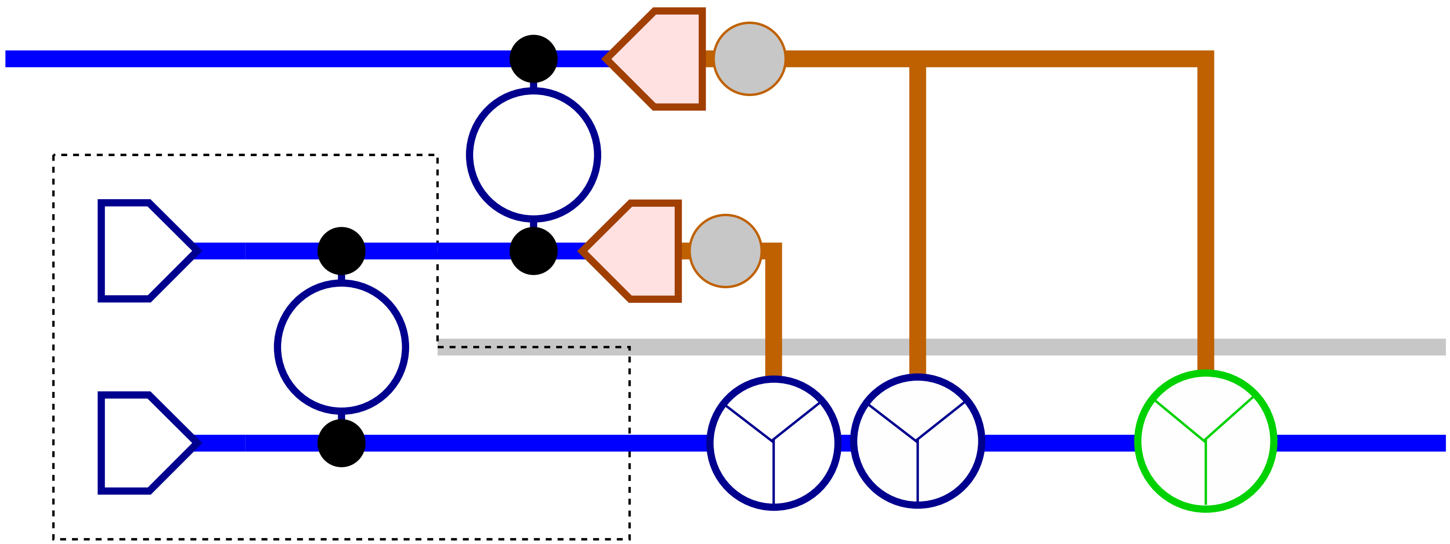
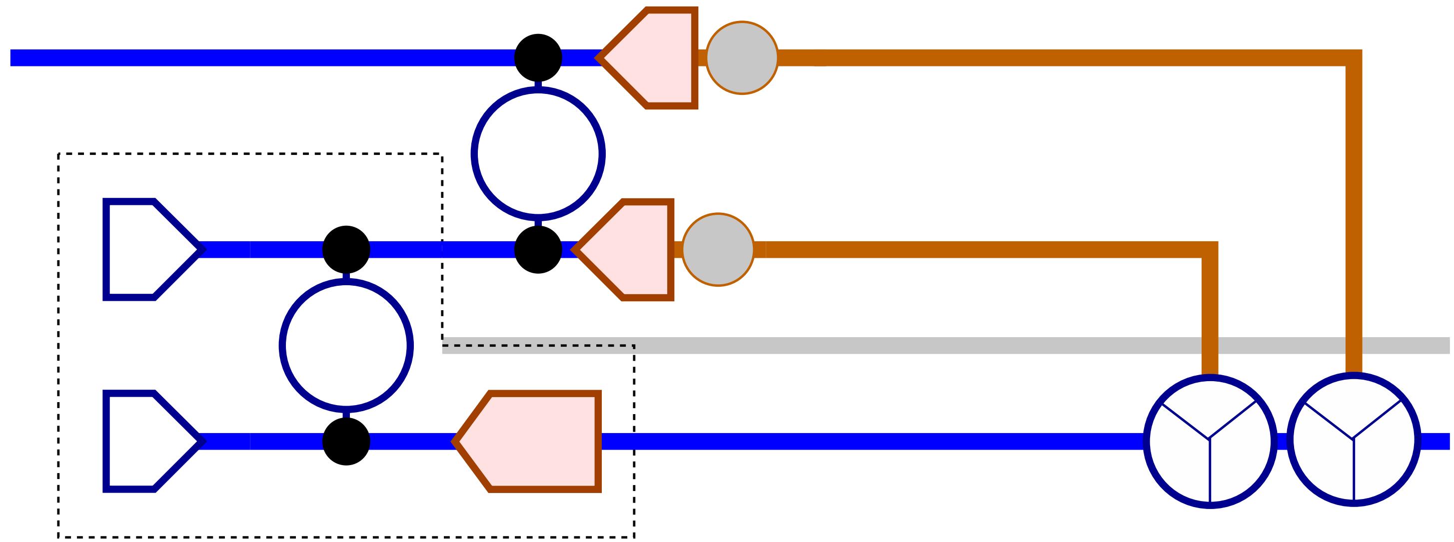
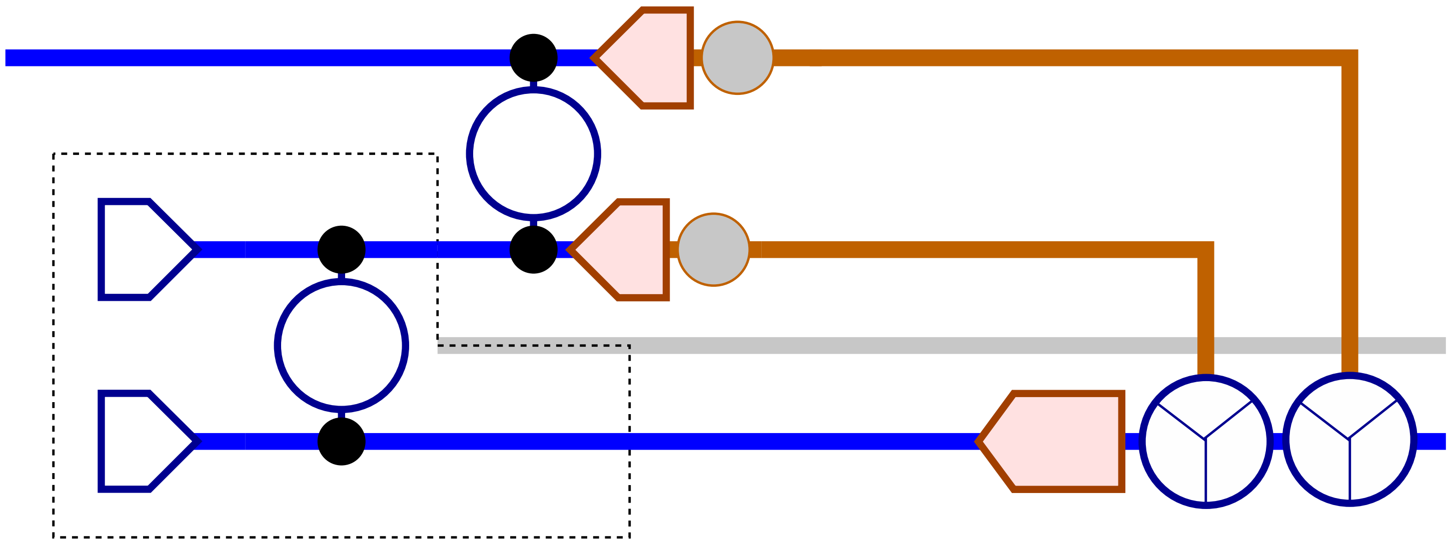


To: Advantages of Teleportation

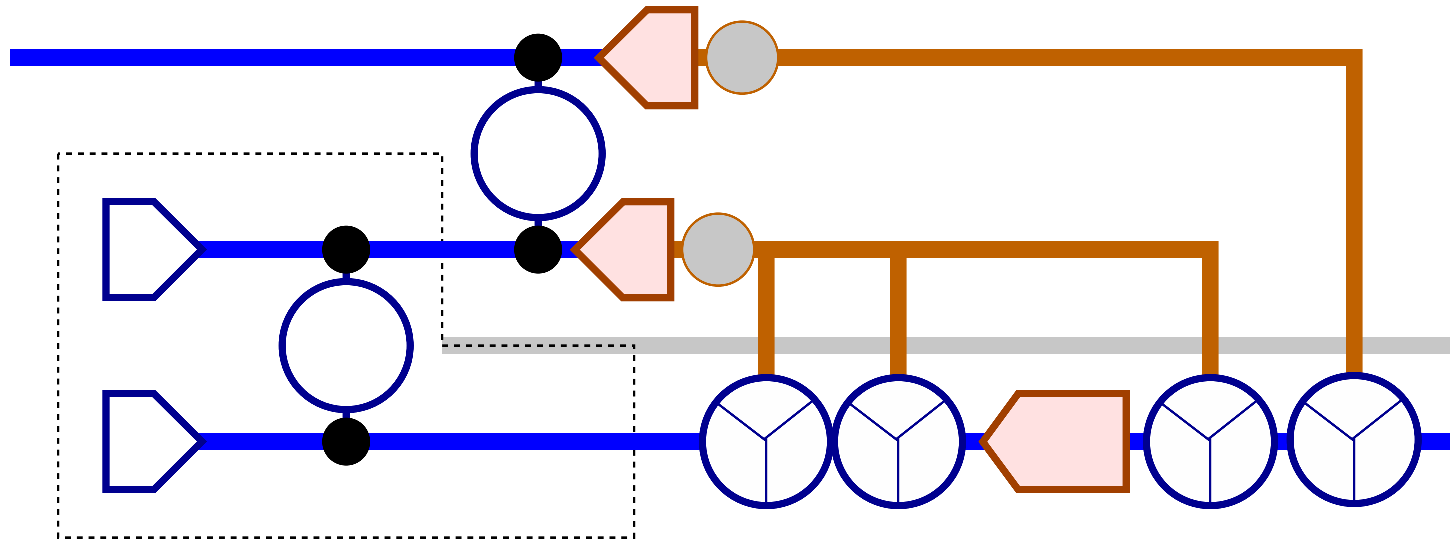
Measurement $=$ Preparation $+$ Teleportation



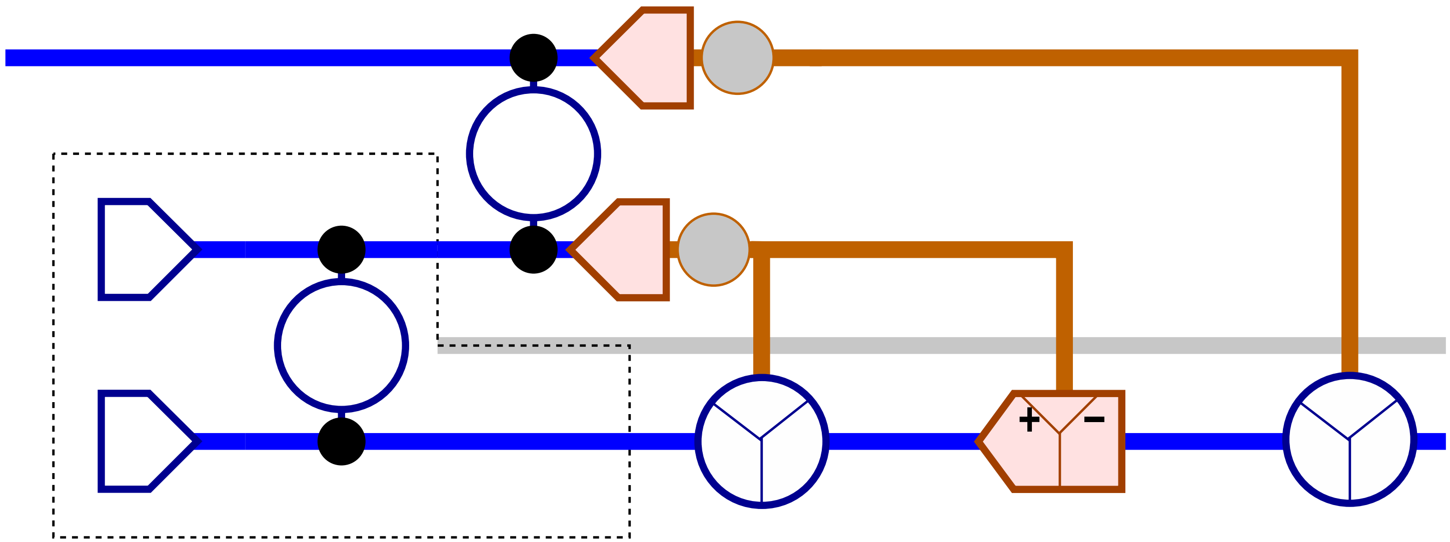
Measurement = Preparation + Teleportation



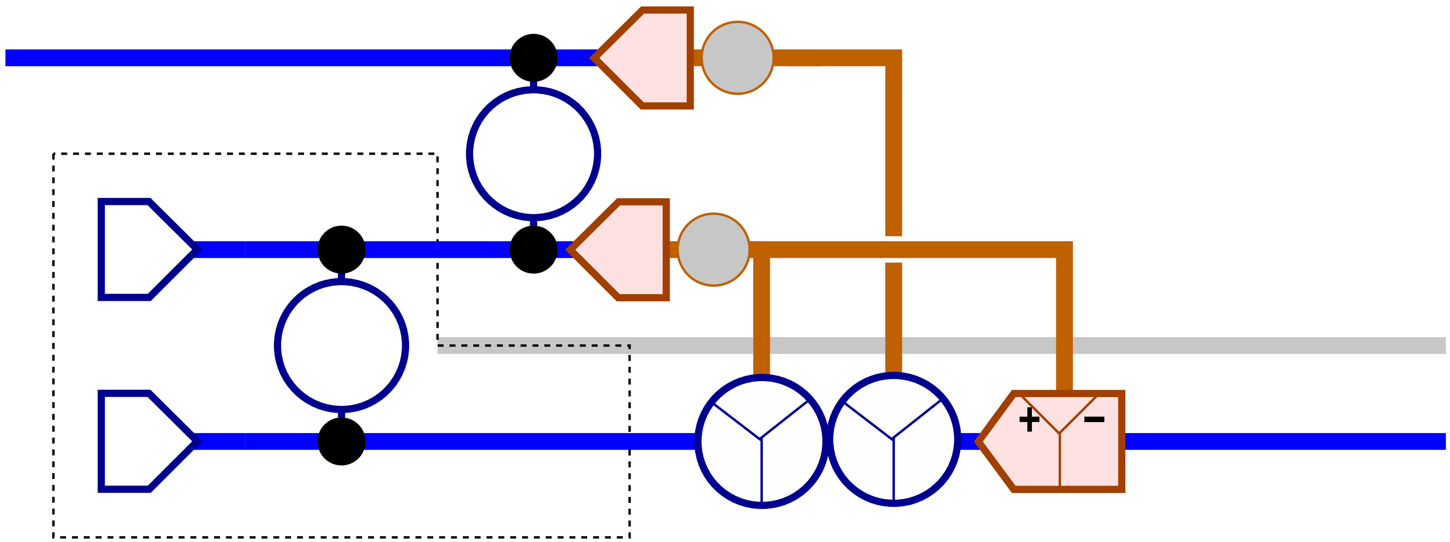
Measurement $=$ Preparation $+$ Teleportation



Measurement $=$ Preparation $+$ Teleportation



Measurement $=$ Preparation $+$ Teleportation



Advantages of Teleportation

- Transversal after successful state preparation.
  - Fault tolerant universality.
  - Robust syndrome detection for recovery from error.
- Good error detection suffices.
  - Reject attempted state preparations if errors are detected.

Back to: Methods for Scalability

Conclusion

- Accuracy threshold questions:
  - Bit flip error model?
  - Erasure error model?
  - Depolarizing error model?

- Worst-case dimension of a maximum size error-detecting quantum code in an n-dimensional space subject to an e-dimensional error model?

  (Best bound known: $\Omega(n/e^2)$.)

References

[1] D. Aharonov and M. Ben-Or. Fault-tolerant quantum computation with constant error. In *Proceedings of the 29th Annual ACM Symposium on the Theory of Computation (STOC)*, pages 176–188, New York, New York, 1996. ACM Press.

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