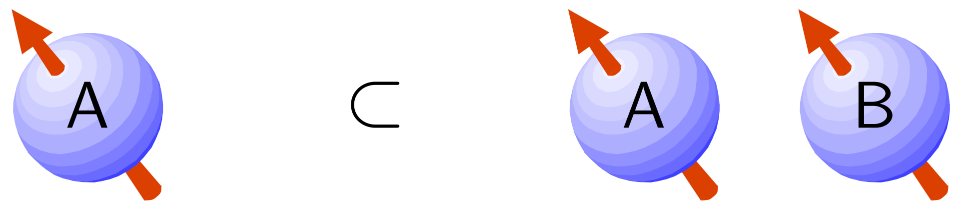
Quantum Error Correction I

Protecting Quantum Information

Manny

- Qubits are subsystems.
- Error control methods.
- Algebraic error models.
- Error detection.
- Error correction.
- Stabilizer codes.

One of Two Qubits



- State spaces:

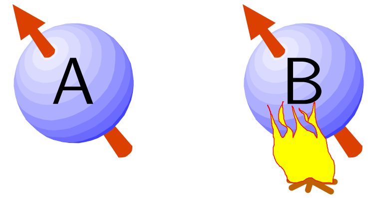
$$\alpha|0\rangle_A + \beta|1\rangle_B \qquad\qquad \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$$

$$\mathcal{Q} \qquad \overset{?}{\subset} \qquad \mathcal{Q} \otimes \mathcal{Q}$$

- Observable algebras:

$$\sigma_x{}^{(A)}, \sigma_y{}^{(A)}, \sigma_z{}^{(A)}, \ldots \quad \subset \quad \sigma_x{}^{(A)}, \ldots, \sigma_x{}^{(B)}, \ldots, \sigma_x{}^{(A)}\sigma_x{}^{(B)}, \ldots$$

Passive Error Control



- Example noise operators:

$$\mathbb{I}, \quad \sigma_x{}^{(\mathrm{B})}, \sigma_y{}^{(\mathrm{B})}, \sigma_z{}^{(\mathrm{B})}.$$

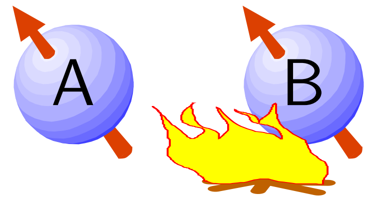
  - Information in A is protected.
  - *No* state of AB is protected.

$$|00\rangle_{\mathrm{AB}} \xrightarrow{\sigma_x{}^{(\mathrm{B})}} |01\rangle_{\mathrm{AB}}$$

  - Observables for A *commute* with errors.

$$\sigma_u{}^{(\mathrm{A})}\sigma_v{}^{(\mathrm{B})} = \sigma_v{}^{(\mathrm{B})}\sigma_u{}^{(\mathrm{A})}.$$

Active Error Control



- Example noise operators:

$$\mathbb{I}, \quad \mathbf{cxnot} = \sigma_x{}^{(\mathsf{B})}\mathbf{cnot}^{(\mathsf{BA})}.$$

  - Start with B in $|0\rangle_{\mathsf{B}}$.

$$|0\rangle_{\mathsf{L}} = |00\rangle_{\mathsf{AB}}, \; |1\rangle_{\mathsf{L}} = |10\rangle_{\mathsf{AB}}$$

  - Information of A preserved after one error.
  - . . . lost after two.

$$|0\rangle_{\mathsf{L}} \overset{\mathbf{cxnot}}{\longrightarrow} |01\rangle_{\mathsf{AB}} \overset{\mathbf{cxnot}}{\longrightarrow} |10\rangle_{\mathsf{AB}} = |1\rangle_{\mathsf{L}}$$

  - Solution: Reset B before errors.

Back to: Error Correction I

Error Control Methods

- Passive error control.
  - Noiseless subsystems.

- Error suppression.
  - Refocusing.
  - Active symmetry enforcement.
  - Decoupling.
  - Reservoir engineering.

- Systematic error control.
  - Rotating frames.
  - Composite pulses.
  - Adiabatic gates.

- Active error control.
  - Periodic error correction.

References

- Prehistory:
  - Quantum Zeno effect: Misra&Sudarshan 1977 [14].
  - Deutsch 1993, Barenco&*al.* 1996 [3].
- Discovery and Theory:
  - Shor 1995 [17], Steane 1995 [19].
  - Bennett&DiVincenzo&Smolin&Wootters 1996 [4], Knill&Laflamme 1996 [10].
  - Calderbank&Shor 1996 [6], Gottesman 1996 [7], Calderbank&Rains&Shor&Sloane 1997 [5].
- Fault tolerance and threshold accuracies:
  - Shor 1996 [18], Kitaev 1997 [9].
  - Aharonov&Ben-Or 1996 [1, 2], Knill&Laflamme&Zurek 1996 [12], Gottesman&Preskill 1997 [8, 16].
- Toward subsystems:
  - Quasi-particles . . .
  - Zanardi&Rasetti 1997 [23], Lidar&Chuang&Whaley 1998 [13].
  - Viola&Knill&Lloyd 1998 [21, 20, 22].
  - Knill&Laflamme 1996 [10], Knill&Laflamme&Viola 2000 [11].

General reference: (M)ike, Ch. 10. Nielsen&Chuang 2001 [15]

The Pauli Error Model

- Error operators:

$$\mathcal{E}_1 \quad = \quad \{\, \mathbb{I},\ {\sigma_x}^{(1)},\ {\sigma_y}^{(1)},\ {\sigma_z}^{(1)},\ {\sigma_x}^{(2)},\ {\sigma_y}^{(2)},\ \ldots\, \}$$

- Weight 2 error operators:

$$\begin{aligned} \mathcal{E}_2 \quad &= \quad \mathcal{E}_1\mathcal{E}_1 \\ &= \quad \{\, \mathbb{I},\ {\sigma_x}^{(1)},\ \ldots\ {\sigma_x}^{(1)}{\sigma_x}^{(2)},\ {\sigma_x}^{(1)}{\sigma_x}^{(3)},\ \ldots\} \end{aligned}$$

- Higher weights:

$$\begin{aligned} \mathcal{E}_k \quad &= \quad \mathcal{E}_1^k = \overbrace{\mathcal{E}_1\mathcal{E}_1\ldots}^{k \text{ times}} \\ \mathcal{E} \quad &= \quad \textstyle\bigcup_{k=1}^{\infty} \mathcal{E}_k \end{aligned}$$

  - The linear span of $\mathcal{E}$ contains all operators.

Algebraic Error Models

- Weight 1 error events:

$$\mathcal{E}_1 \quad = \quad \{\, \mathbb{I},\ E_1,\ E_2,\ \ldots\}$$

- Weight k error events:

$$\mathcal{E}_k \quad = \quad \mathcal{E}_1^k = \overbrace{\mathcal{E}_1 \mathcal{E}_1 \ldots}^{k \text{ times}}$$

- Error algebra:

$$\mathcal{E} \quad = \quad \mathrm{span} \bigcup_{k=1}^{\infty} \mathcal{E}_k$$

Flip Errors

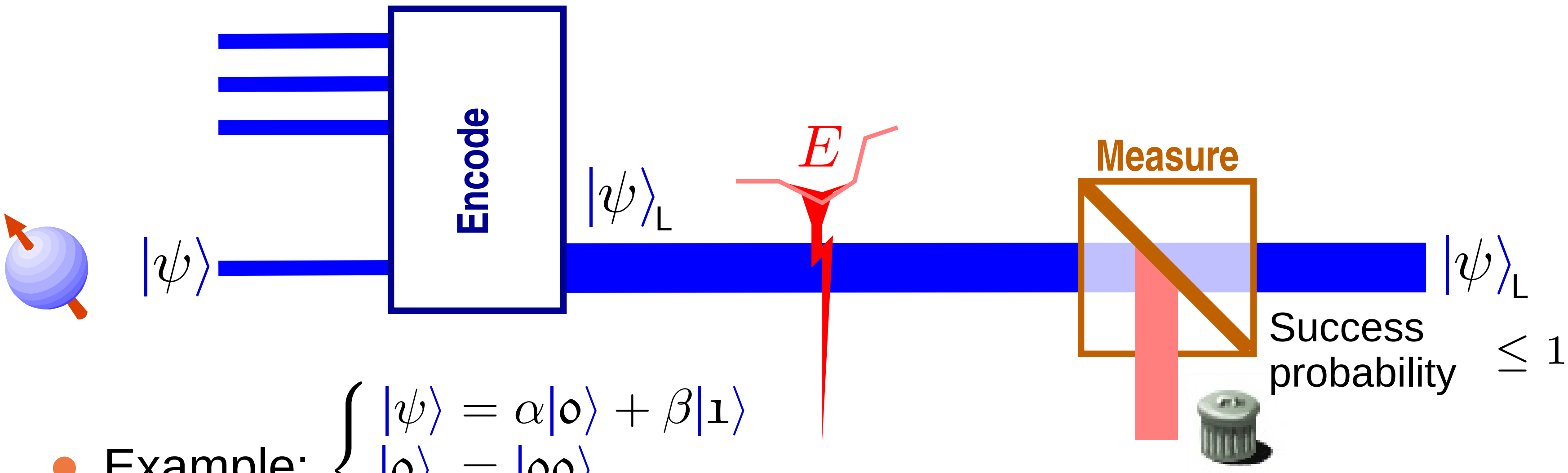
$$\mathcal{E}_1 = \left\{ \mathbb{I},\ {\sigma_x}^{(1)},\ {\sigma_x}^{(2)},\ {\sigma_x}^{(3)},\ \ldots \right\}$$

- Classical errors $+$ superposition principle.
  - Examples.

$$\begin{aligned}
|0010\rangle &\xrightarrow{{\sigma_x}^{(2)}} |0110\rangle \\
|0010\rangle &\xrightarrow{{\sigma_x}^{(1)}{\sigma_x}^{(3)}} |1000\rangle \\
\frac{1}{\sqrt{2}}\Big(|0010\rangle + |0011\rangle\Big) &\xrightarrow{{\sigma_x}^{(4)}} \frac{1}{\sqrt{2}}\Big(|0011\rangle + |0010\rangle\Big) \\
&= \frac{1}{\sqrt{2}}\Big(|0010\rangle + |0011\rangle\Big)
\end{aligned}$$

- Size of common eigenspaces of ${\sigma_x}^{(\mathrm{i})}$?

Error Detection I



- Example: $\begin{cases} |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \\ |0\rangle_L = |00\rangle \\ |1\rangle_L = |11\rangle \end{cases}$

$$|00\rangle\langle 00| + |11\rangle\langle 11|$$

$$\alpha|00\rangle + \beta|11\rangle \quad \xrightarrow{\mathbb{I}^{(12)}} \quad \alpha|00\rangle + \beta|11\rangle$$

$$\xrightarrow{{\sigma_x}^{(1)}} \quad \alpha|10\rangle + \beta|01\rangle$$

$$\xrightarrow{{\sigma_x}^{(2)}} \quad \alpha|01\rangle + \beta|10\rangle$$

$$\xrightarrow{{\sigma_x}^{(1)}{\sigma_x}^{(2)}} \quad \alpha|11\rangle + \beta|00\rangle$$

Error Detection II

- A *quantum code* is a subspace $\mathcal{C}$.
  - Projection operator: $P_{\mathcal{C}}$.
  - Logical basis: $|0\rangle_{\mathrm{L}}, |1\rangle_{\mathrm{L}}, |2\rangle_{\mathrm{L}}, \ldots$.
- $\mathcal{C}$ *detects* E if

$$P_{\mathcal{C}} E P_{\mathcal{C}} = \lambda_E P_{\mathcal{C}}$$

- Equivalently:

$$E = \left(\begin{array}{cc} \begin{array}{cc} & \overbrace{}^{\mathcal{C}} \\ \mathcal{C}\left\{\right. & \begin{array}{cccc} \lambda_E & 0 & \ldots & 0 \\ 0 & \lambda_E & & \vdots \\ \vdots & & \ddots & \\ 0 & & \ldots & \lambda_E \end{array} \end{array} & E_{12} \\ E_{21} & E_{22} \end{array}\right)$$

Error Detection II

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$$P_{\mathcal{C}} E P_{\mathcal{C}} = \lambda_E P_{\mathcal{C}}$$

- Equivalently: For all $|\phi\rangle_{\mathsf{L}}$

$${}_{\mathsf{L}}\langle\phi|E|\phi\rangle_{\mathsf{L}} = \lambda_E$$

- Equivalently: For all $|\phi\rangle_{\mathsf{L}}, |\psi\rangle_{\mathsf{L}}$

$$|\phi\rangle_{\mathsf{L}} \perp |\psi\rangle_{\mathsf{L}} \Rightarrow E|\phi\rangle_{\mathsf{L}} \perp |\psi\rangle_{\mathsf{L}}$$

Minimum Distance I

- $\mathcal{C}$ has *minimum distance* $\geq d$ if

  $\mathcal{C}$ detects all errors of weight $d-1$.

- $\mathcal{C} = \mathrm{span}\left(|00\rangle, |11\rangle \right)$ is a $[[2,1,2]]_{\sigma_x^{(i)}}$ code.

- $\mathcal{C}$ is a $[[n,k,d]]_{\mathcal{E}_1}$ code means:
  - *Length* n: Total number of qubits is n.
  - k encoded qubits, $\dim \mathcal{C} = 2^k$.
  - Minimum distance at least d for $\mathcal{E}_1$.

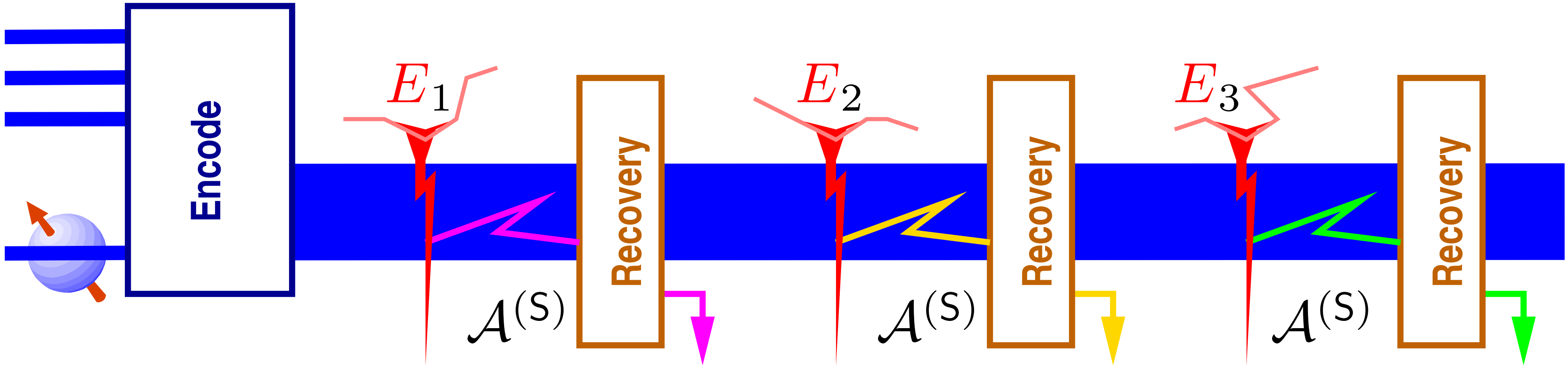
Minimum Distance II

- Construct a $[[3,1,3]]_{\sigma_x^{(i)}}$ code — greedily:
  - Add: $|0\rangle_L = |000\rangle$.
  - $|1\rangle_L$ must be orthogonal to

$$|000\rangle$$
$$|100\rangle,\ |010\rangle,\ |001\rangle$$
$$|110\rangle,\ |101\rangle,\ |011\rangle$$

  - Choose $|1\rangle_L = |111\rangle$.
  - Encode $\alpha|0\rangle + \beta|1\rangle \to \alpha|000\rangle + \beta|111\rangle$.
- . . . the three bit repetition code.

Error Correction Process



- Where is the encoded qubit?
  - Observable algebra always defined:

$$\mathcal{A}_t = \mathrm{span}\left(\mathbb{I}^{(\mathsf{S}_t)},\ {\sigma_x}^{(\mathsf{S}_t)},\ {\sigma_y}^{(\mathsf{S}_t)},\ {\sigma_z}^{(\mathsf{S}_t)}\right)$$

  - $\mathcal{A}^{(\mathsf{S})}$: Algebra between errors and recovery.

Subsystems

- A *subsystem* is a factor of a subspace of $\mathcal{H}$.
- Specifying a subsystem S:
  - Decompose: $\mathcal{H} \simeq \left(\mathcal{S} \otimes \mathcal{T}\right) \oplus \mathcal{R}$
  - Observables: $*$-algebra $\mathcal{A}^{(\mathsf{S})} \simeq \mathrm{Matrices}(\dim \mathcal{S})$.
- Example with two qubits:
  - Decompose:

$$|0\rangle_{\mathsf{S}}|0\rangle_{\mathsf{T}} = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big), \quad |0\rangle_{\mathsf{S}}|1\rangle_{\mathsf{T}} = \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big)$$

$$|1\rangle_{\mathsf{S}}|0\rangle_{\mathsf{T}} = \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big), \quad |1\rangle_{\mathsf{S}}|1\rangle_{\mathsf{T}} = \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big)$$

  - Observables: $\left\{ \begin{array}{lcl} {\sigma_x}^{(\mathsf{S})} & = & {\sigma_x}^{(2)} \\ {\sigma_z}^{(\mathsf{S})} & = & {\sigma_z}^{(1)}{\sigma_z}^{(2)} \end{array} \right.$

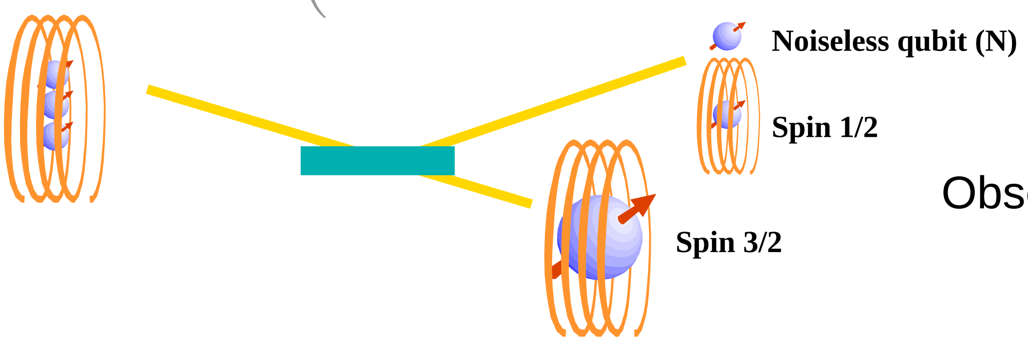
Noiseless subsystems

- Subsystem S is *noiseless* for $\mathcal{E}$ if $[\mathcal{E}, \mathcal{A}^{(\mathsf{S})}] = 0$.
- $\mathcal{E} = \mathrm{span}\left(\mathbb{I}, {\sigma_x}^{(1)}{\sigma_x}^{(2)}, {\sigma_z}^{(1)}\right)$:
  - Observables: $\left\{\begin{array}{lcl} {\sigma_x}^{(\mathsf{S})} & = & {\sigma_x}^{(2)} \\ {\sigma_z}^{(\mathsf{S})} & = & {\sigma_z}^{(1)}{\sigma_z}^{(2)} \end{array}\right.$
  - Decompose:

$$|0\rangle_{\mathsf{S}}|0\rangle_{\mathsf{T}} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \quad |0\rangle_{\mathsf{S}}|1\rangle_{\mathsf{T}} = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

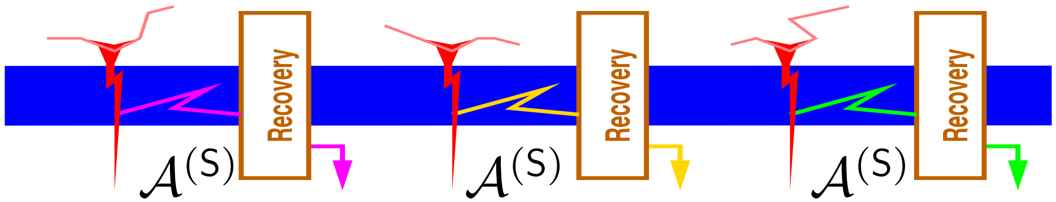
$$|1\rangle_{\mathsf{S}}|0\rangle_{\mathsf{T}} = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right), \quad |1\rangle_{\mathsf{S}}|1\rangle_{\mathsf{T}} = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

- $\mathcal{E} = \mathrm{span}\left(\mathbb{I}, {\sigma_x}^{(1)}+{\sigma_x}^{(2)}+{\sigma_x}^{(3)}, {\sigma_y}^{(1)}+{\sigma_y}^{(2)}+{\sigma_y}^{(3)}, {\sigma_z}^{(1)}+{\sigma_z}^{(2)}+{\sigma_z}^{(3)}\right)$:



Observables: $\left\{\begin{array}{l} {\sigma_U}^{(\mathsf{N})} = \sigma^{(\mathsf{A})} \cdot \sigma^{(\mathsf{B})} \\ {\sigma_V}^{(\mathsf{N})} = \sigma^{(\mathsf{A})} \cdot \sigma^{(\mathsf{C})} \end{array}\right.$

Error Correcting Subsystems I



- Protection requires:

$$[ER, \mathcal{A}^{(\mathsf{S})}] = 0$$

R: Any operator of the recovery.

- *Error correcting subsystem* $(\mathsf{S}, |0\rangle_{\mathsf{T}})$:

$$\mathcal{H} \simeq \left(\mathcal{S} \otimes \mathcal{T}\right) \oplus \mathcal{R}, \quad |0\rangle_{\mathsf{T}}$$

- Usage: 1. Reject R. 2. Reset T to $|0\rangle_{\mathsf{T}}$. 3. Wait . . .

- $(\mathsf{S}, |0\rangle_{\mathsf{T}})$ *corrects* E if

$$E|\psi\rangle_{\mathsf{S}}|0\rangle_{\mathsf{T}} = |\psi\rangle_{\mathsf{S}}|\phi_E\rangle_{\mathsf{T}}$$

Example.

Error Correcting Subsystems II

- The three qubit $[[3,1,3]]_{\sigma_x{}^{(\mathrm{i})}}$ repetition code.

$$\mathcal{C} = \mathrm{span}\left(|000\rangle,\ |111\rangle\right)$$

- As a subsystem:

$$\begin{array}{rcl} |0\rangle_{\mathsf{S}}|0\rangle_{\mathsf{T}} & = & |000\rangle \\ \\ |0\rangle_{\mathsf{S}}|1\rangle_{\mathsf{T}} & = & |100\rangle \\ |0\rangle_{\mathsf{S}}|2\rangle_{\mathsf{T}} & = & |010\rangle \\ |0\rangle_{\mathsf{S}}|3\rangle_{\mathsf{T}} & = & |001\rangle \end{array} \qquad \begin{array}{rcl} |1\rangle_{\mathsf{S}}|0\rangle_{\mathsf{T}} & = & |111\rangle \\ \\ |1\rangle_{\mathsf{S}}|1\rangle_{\mathsf{T}} & = & |011\rangle \\ |1\rangle_{\mathsf{S}}|2\rangle_{\mathsf{T}} & = & |101\rangle \\ |1\rangle_{\mathsf{S}}|3\rangle_{\mathsf{T}} & = & |110\rangle \end{array}$$

- Detection property:

$$\begin{array}{rcl} P_{\mathcal{C}}\sigma_x{}^{(\mathrm{i})\dagger}\sigma_x{}^{(\mathrm{j})}P_{\mathcal{C}} & = & \delta_{ij}P_{\mathcal{C}} \\ \\ P_{\mathcal{C}} & = & |0\rangle_{\mathsf{T}}{}^{\mathsf{T}}\langle 0| \end{array}$$

From Correction to Detection

- Suppose $(\mathsf{S}, |0\rangle_{\mathsf{T}})$ corrects errors in $\mathcal{E}$.

  Define $\mathcal{C}$ by $P_{\mathcal{C}} = |0\rangle_{\mathsf{T}}{}^{\mathsf{T}}\langle 0|$.
  Then $\mathcal{C}$ detects every error in $\mathcal{E}^\dagger\mathcal{E}$.

- Proof:

$$
\begin{aligned}
{}^{\mathsf{S}}\langle\psi|{}^{\mathsf{T}}\langle 0|E^\dagger D|0\rangle_{\mathsf{T}}|\psi\rangle_{\mathsf{S}} &= {}^{\mathsf{S}}\langle\psi|{}^{\mathsf{T}}\langle 0|E^\dagger\ |\phi(D)\rangle_{\mathsf{T}}\ |\psi\rangle_{\mathsf{S}} \\
&= {}^{\mathsf{S}}\langle\psi|\ {}^{\mathsf{T}}\langle\phi(E)|\ |\phi(D)\rangle_{\mathsf{T}}\ |\psi\rangle_{\mathsf{S}} \\
&= {}^{\mathsf{S}}\langle\psi||\psi\rangle_{\mathsf{S}}\ {}^{\mathsf{T}}\langle\phi(E)||\phi(D)\rangle_{\mathsf{T}} \\
&= {}^{\mathsf{T}}\langle\phi(E)||\phi(D)\rangle_{\mathsf{T}} \\
&= \lambda_{E^\dagger D}
\end{aligned}
$$

From Detection to Correction

- Suppose $\mathcal{C}$ detects errors in $\mathcal{E}^\dagger\mathcal{E}$.
  There exists $(\mathsf{S}, |0\rangle_{\mathsf{T}})$ such that for some unitary U:

$$U^\dagger P_{\mathcal{C}} U = |0\rangle_{\mathsf{T}}{}^{\mathsf{T}}\langle 0|$$

$(\mathsf{S}, |0\rangle_{\mathsf{T}})$ corrects $\mathrm{span}(\mathcal{E}U)$.

- Proof:
  - $\mathrm{span}(\mathcal{E}) = \mathrm{span}(E_0, E_1, \ldots)$.
  - $P_{\mathcal{C}} E_i^\dagger E_j P_{\mathcal{C}} = \lambda_{ij} P_{\mathcal{C}}$.
  - $\Lambda = (\lambda_{ij})_{ij}$ Hermitian $\Rightarrow$ change basis of $\mathrm{span}(\mathcal{E})$:

$$\lambda_{ij} = \delta_{ij}.$$

  - Let $|\psi\rangle_{\mathsf{L}}$ be a state of $\mathcal{C}$. Define

$$|\psi\rangle_{\mathsf{S}}|i\rangle_{\mathsf{T}} \quad = \quad E_i|\psi\rangle_{\mathsf{L}}$$

  - Choose U unitary: $U|\psi\rangle_{\mathsf{S}}|0\rangle_{\mathsf{T}} = |\psi\rangle_{\mathsf{L}}$.
  - Compute: $E_i U|\psi\rangle_{\mathsf{S}}|0\rangle_{\mathsf{T}} = E_i|\psi\rangle_{\mathsf{L}} = |\psi\rangle_{\mathsf{S}}|i\rangle_{\mathsf{T}}$

QEC Theorems

Theorem: Given a quantum system. There exists an error-correcting subsystem for $\mathcal{E}$ iff there exists a $\mathcal{E}^\dagger\mathcal{E}$ detecting quantum code.

Corollary: Assume $\mathcal{E}_1^\dagger = \mathcal{E}_1$. If $\mathcal{C}$ has minimum distance $2e+1$, then $\mathcal{C}$ induces an e-error-correcting subsystem.

- Coding theory lingo: A $[[n, k, 2e+1]]$ code is e-error correcting.

Stabilizer Codes I

- Conventions:

$$X = \sigma_x,\ Y = \sigma_y,\ Z = \sigma_z$$
$$IXIII = {\sigma_x}^{(2)},\ IIYZI = {\sigma_y}^{(3)}{\sigma_z}^{(4)}$$

- *Pauli group*: Products of σ_u up to sign.
- Stabilizer of $\mathcal{C} = \mathrm{span}\big(|000\rangle, |111\rangle\big)$:

$$\{III, ZZI, IZZ, ZIZ\} = \langle\ ZZI, IZZ\ \rangle$$

- Properties:

$$\mathcal{C} \quad \text{detects} \quad IXI$$
$$IXI \cdot ZZI \quad = \quad -ZZI \cdot IXI$$

Stabilizer Codes II

- A *stabilizer code (Pauli code, symplectic code)* is a common eigenspace of a commutative subgroup N of the Pauli group.
- Define
  $$N^{\perp} = \{\text{Pauli operators which commute with } N\}.$$

Theorem: A stabilizer code of N detects all Pauli operators except those in $N^{\perp} \setminus N$.

- Proof: For $\rho \in N$, let $\lambda(\rho)$ be the common eigenvalue.
  - $\sigma \in N$: ok.
  - $\sigma \notin N^{\perp}$: Choose $\rho \in N$, $\sigma\rho = -\rho\sigma$. For $|\psi\rangle$ in the code:
    $$\begin{aligned} \rho|\psi\rangle &= \lambda(\rho)|\psi\rangle \\ \rho\,\sigma|\psi\rangle &= -\sigma\,\rho|\psi\rangle \\ &= -\lambda(\rho)\sigma|\psi\rangle \qquad \ldots\sigma|\psi\rangle \text{ is orthogonal to the code.} \end{aligned}$$

The 5 Qubit Code

- Minimum distance 3 code for Pauli errors:
  Stabilizer: $\langle\, YZZYI, IYZZY, YIYZZ, ZYIYZ \,\rangle$.
- Example distance check:
  - Every weight ≤ 2 Pauli product $UVIII$ is detected:

$$\begin{array}{|cc|ccc}
Y & Z & Z & Y & I \\
I & Y & Z & Z & Y \\
Y & I & Y & Z & Z \\
Z & Y & I & Y & Z
\end{array}$$

  - Restrict to first two columns:
    $\langle\, YZ, IY, YI, ZY \,\rangle \ldots$ generates all Pauli products.
  - $\Rightarrow$ every non-identity $UVIII$ anticommutes with a stabilizer.
- Rules:

$$\begin{array}{ccccccc}
X \cdot Y & \sim & Z & & 01 \oplus 10 & = & 11 \\
Y \cdot Z & \sim & X & \Leftrightarrow & 10 \oplus 11 & = & 01 \\
Z \cdot X & \sim & Y & & 11 \oplus 01 & = & 10
\end{array}$$

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